Existence as Positive Measurement: Mass Nouns and Intensionality

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Abstract

Focusing particularly on mass nouns, we develop an ontology for natural language semantics based on measure theory. The new framework makes for rigorous discussion on changes of states and on events of continuous nature.

1 Introduction

In languages like English, nouns come in two kinds: count and mass. Count nouns, such as *child* and *statue*, exhibit a singular/plural morphological distinction, and typically denote things or individuated objects. In contrast, mass nouns, such as *milk* and *clay*, have no morphological number distinction, and typically denote stuffs or materials.

Two properties standardly associated with mass nouns are **cumulativity** (Quine, 1960) and **distributivity** (Cheng, 1973). Cumulativity refers to the fact that if x is milk and y is milk, then the sum of x and y is also milk. Distributivity refers to the fact that any part of milk is also milk. Theoretical analyses of mass nouns that take these properties into account are naturally led to assume some kind of Boolean structure that need not be atomic (Bunt, 1979; Link, 1983; Roeper, 1983; Lønning, 1987). Let's follow suit and assume that mass entities forms a nonatomic Boolean algebra $\langle D, \vee, \wedge, \neg, 0_D, 1_D \rangle$, where $\vee$ is join (or supremum), $\wedge$ is meet (or infimum), $\neg$ is complement, and 0_D and 1_D are the bottom and top elements. It is reasonable to take this algebra to be σ**-complete**, meaning that every countable subset has a join and a meet. The part-of relation $\leq$ on D is the partial order defined by

$$x \leq y \quad \text{iff} \quad x \vee y = y \quad (\text{iff} \quad x \wedge y = x).$$

For more on Boolean algebras, the reader is referred to Givant and Halmos (2009).

Assuming that the denotation of *milk* is identical to that of the predicate *is milk*, it seems appropriate to have

(1) $[\![\text{milk}]\!] = \{\, x \in D \mid x \leq m \,\}$,

where m stands for the sum of all milk in the world. To use a technical term, $[\![\text{milk}]\!]$ is an ideal.

Definition. A subset I of a Boolean algebra $\mathscr{B}$ is an **ideal** iff it satisfies the following conditions:

(C1) $0_{\mathscr{B}} \in I$.

(C2) If $x \in I$ and $y \in I$, then $x \vee y \in I$.

(C3) If $x \in I$ and $y \leq x$, then $y \in I$.

One can see that (C2) and (C3) correspond to cumulativity and distributivity, respectively. (C1) is to ensure that I is nonempty, but in the present context, it implies that $0_D \in [\![\text{milk}]\!]$. This might appear nonsensical, but I assume it to be a mere technicality with no real semantic consequences. An ideal I is said to be σ**-complete** if (C2) is strengthened to the following:

(C2′) If $S \subseteq I$ is countable, then $\bigvee S \in I$.

The elements that are less than or equal to some element $a \in \mathscr{B}$ form an ideal, known as the **principal ideal** generated by a, and written $\downarrow a$. With this notation, (1) can be expressed as $[\![\text{milk}]\!] = \downarrow m$. It is easy to see that a principal ideal in a σ-complete algebra is necessarily a σ-complete ideal.

This kind of classic model, which is static in nature, might be good enough, if all that there is is what exists at this moment in actuality. However, the existing entities might not exist in the past or future, and can be hypothesized to be absent in a state of affairs—or possible world—other than actuality. Conversely, something that does not exist now may exist at a different time or in a different world. As language enables such intensional talk, a simple analysis like (1) is bound to be deficient.

This paper develops a theory to repair this inadequacy. Section 2 explains our ontological standpoint that equates existence with positive measurement. Section 3 defines the life of an entity, the time period throughout which the entity exists. This leads us to review our ontological assumptions about times, which shall now be understood as (equivalence classes of) sets of time points of positive measure. Section 4 looks into the denotation of a mass noun in our new ontological setting. Section 5 examines sentences of continuous production or consumption and concludes that they call for mathematical integration, which shall be carried out measure-theoretically. Finally, Section 6 notes connections between telicity and integration.

2 To Be Is To Measure Positively

Quantities of entities denoted by count nouns are numerically expressed with cardinal numbers, as in *9 children*. In contrast, entities denoted by mass nouns are generally not countable, and their quantities are numerically expressed with measuring expressions such as *9.8 liters of*. To analyze them under the assumption that mass entities form a σ-complete Boolean algebra, functions known as measures come in handy (Halmos, 1950; Cartwright, 1975; Krifka, 1989; Higginbotham, 1994).

Definition. A **measure** μ on a σ-complete Boolean algebra $\mathscr{B}$ is a function from $\mathscr{B}$ into $\mathbb{R} \cup \{\infty\}$ that satisfies the following three conditions:

- Nonnegativity: $\mu(x) \geq 0$ for all x.
- $\mu(0_{\mathscr{B}}) = 0$.
- Countable additivity: if $\{x_n\}_{n \in \mathbb{N}}$ is a sequence of pairwise disjoint elements (i.e. $i \neq j$ implies $x_i \wedge x_j = 0_{\mathscr{B}}$), then $\mu\left(\bigvee_{n \in \mathbb{N}} x_n\right) = \sum_{n \in \mathbb{N}} \mu(x_n)$.

μ is called a **positive measure** if it further satisfies

- Positivity: $\mu(x) > 0$ for all $x > 0_{\mathscr{B}}$.

A Boolean algebra equipped with a positive measure is called a **measure algebra**.

Lemma 1. *If $x \leq y$, then $\mu(x) \leq \mu(y)$.*

Proof. If $x \leq y$, then $x \wedge y = x$, so $y = 1_D \wedge y = (x \vee \neg x) \wedge y = (x \wedge y) \vee (\neg x \wedge y) = x \vee (\neg x \wedge y)$, so $\mu(y) = \mu(x \vee (\neg x \wedge y)) = \mu(x) + \mu(\neg x \wedge y) \geq \mu(x)$ by countable additivity and nonnegativity. □

Using measures, we may expect sentences asserting the existence of mass entities as in (2) to translate as conditions that those entities have a positive measurement as in (3):

(2) a. There is some milk in the tank.
b. There is 9.8 liters of milk in the tank.

(3) a. $\mu(\bigvee [\![\text{milk in the tank}]\!]) > 0$.
b. $\mu_{\text{liter}}(\bigvee [\![\text{milk in the tank}]\!]) \geq 9.8$.[1]

However, such translations give us a conundrum. Unless μ is a positive measure, it is possible that $a = \bigvee([\![\text{milk in the tank}]\!]) \neq 0_D$ and yet $\mu(a) = 0$. If this happens, then (3-a) predicts (2-a) to be false even though the entity a *is* milk in the tank.

To avoid such puzzling situations, one might stipulate that μ be a positive measure. This is not what we want, however. In the previous section, we have noted that things change, and that something that exists now might be no more at other times or in other worlds, and vice versa. I propose that we capture this intuition by employing a family $\{\mu^{w,p}\}_{w \in W, p \in T}$ of measures, where W is the set of possible worlds and T the set of time points. Our slogan here is "existence as positive measurement," which means something like the following:

(4) An entity $x \in D$ exists at p in w iff $\mu^{w,p}(x) > 0$.

This will allow us to describe how entities come into or go out of existence. For instance, if x was born at noon, then $\mu^{w,p}(x) = 0$ if p is a time point before the noon, and $\mu^{w,p}(x) > 0$ if p is a time point after the noon. On this view, entities fundamentally never come into or go out of existence. They are always there as elements of D and may or may not have a positive measurement, depending on the world and the time. On this approach, it is essential that $\mu^{w,p}$ be not a positive measure.

3 The Life of an Entity

Let's discuss the life of a mass entity x, i.e., the set of time points at which x exists.[2] In what follows,

[1] We have "$\geq$" rather than "=" here because (2-b) is consistent with there being more than 9.8 liters of milk in the tank. The impression that (2-b) says that there is exactly 9.8 liters of milk in the tank can be attributed to scalar implicature.

[2] By "a mass entity x," I mean an entity that happens to be predicated of by a mass noun. I am not sure whether the count/mass distinction is inherent in the entities in the model. It seems possible for an entity to be predicated of by a count noun and by a mass noun at the same time, as in *This is furniture—a chair to be precise.*

we will not vary the world parameter and focus on a single, fixed world. Accordingly, we will drop the world parameter and simply write μ^p. Under the slogan "existence as positive measurement," that x exists at p will mean that x measures positively at p. However, if some nonzero part of x (i.e., part of x that is not 0_D) is of measure zero at p, we do not want to think that x exists at p. In order to be able to assert that x exists at p, every bit of x must measure positively. We therefore put forward the following definition.

Definition. The **life of** x **with respect to** a family $\{\mu^p\}_{p\in T}$ of measures, written $\heartsuit_\mu(x)$, is

$$\heartsuit_\mu(x) := \{\, p \in T \mid \forall y(0_D < y \leq x \to \mu^p(y) > 0)\,\}.$$

Lemma 2. (i) $\heartsuit_\mu(\bigvee X) = \bigcap_{x\in X} \heartsuit_\mu(x)$ *for all countable* $X \subseteq D$.

(ii) *If* $x \leq y$*, then* $\heartsuit_\mu(y) \subseteq \heartsuit_\mu(x)$ *for all* $x, y \in D$.

Proof. (i) For every $x \in X$, since $x \leq \bigvee X$, we have

$$\begin{aligned} & p \in \heartsuit_\mu(\textstyle\bigvee X) \\ \iff & \forall y(0_D < y \leq \textstyle\bigvee X \to \mu^p(y) > 0) \\ \implies & \forall y(0_D < y \leq x \to \mu^p(y) > 0) \\ \iff & p \in \heartsuit_\mu(x), \end{aligned}$$

so $\heartsuit_\mu(\bigvee X) \subseteq \heartsuit_\mu(x)$. Hence $\heartsuit_\mu(\bigvee X) \subseteq \bigcap_{x\in X} \heartsuit_\mu(x)$.

Conversely, suppose that $p \in \bigcap_{x\in X} \heartsuit_\mu(x)$. Suppose that $0_D < y \leq \bigvee X$. Then $\bigvee_{x\in X}(y \wedge x) = y \wedge (\bigvee X) = y > 0_D$, so there is some $x_0 \in X$ with $y \wedge x_0 \neq 0_D$. Since $p \in \heartsuit_\mu(x_0)$ and $y \wedge x_0 \leq x_0$, we have $\mu^p(y \wedge x_0) > 0$ by the definition of $\heartsuit_\mu$. Since $\mu^p(y) \geq \mu^p(y \wedge x_0)$ by Lemma 1, it follows that $\mu^p(y) > 0$. This shows that $p \in \heartsuit_\mu(\bigvee X)$.

(ii) If $x \leq y$, then $x \vee y = y$, so by (i), $\heartsuit_\mu(y) = \heartsuit_\mu(x \vee y) = \heartsuit_\mu(x) \cap \heartsuit_\mu(y) \subseteq \heartsuit_\mu(x)$. □

It might seem that the assertion that an entity x exists sometime will be translated as $\heartsuit_\mu(x) \neq \varnothing$. However, what if $\heartsuit_\mu(x)$ consists of a single time point p_0, i.e., $\heartsuit_\mu(x) = \{p_0\}$? This means that x had some physical presence at p_0, but none either before or after. Do we still want to say that x existed? Even if such a situation obtained in reality, there would be no way of ascertaining it, and I believe most of us intuitively think that things do not work that way. Then, we might as well ignore the possibility of such literally instantaneous existence. For something to exist, it must be there for some positive length of time. It is time we extended the slogan "existence as positive measurement" to the realm of times.

As in physics, let's identify T with the set $\mathbb{R}$ of real numbers. Then, the **Lebesgue measure** μ_{L} can be used to measure the length of a subset of T. For instance, if an interval $X \subseteq T$ begins at p and ends at q, then $\mu_{\mathrm{L}}(X) = q - p$ (regardless of whether X is open, closed, or half-open). Let $\mathscr{L}(T)$ be the set of Lebesgue-measurable subsets of T, i.e., subsets X of T for which $\mu_{\mathrm{L}}(X)$ is defined. It is known that $\langle \mathscr{L}(T), \cup, \cap, {}^{\mathsf{c}}, \varnothing, T\rangle$ is a σ-complete Boolean algebra, and μ_{L} gives a measure on it. Since $\mu_{\mathrm{L}}(\{p_0\}) = 0$, through the lens of the Lebesgue measure, $\{p_0\}$ is equivalent to $\varnothing$. In other words, being there only at p_0 will be identified with being there at no time at all. The mathematical way to achieve this perspective is to go to the **quotient algebra** of $\mathscr{L}(T)$ modulo the set I of sets of measure zero, i.e.,

$$I = \{\, X \in \mathscr{L}(T) \mid \mu_{\mathrm{L}}(X) = 0\,\}.$$

I is a σ-complete ideal in $\mathscr{L}(T)$, and one can define an equivalence relation $\sim$ on $\mathscr{L}(T)$ by

$$X \sim Y \quad \text{iff} \quad (X \cap Y^{\mathsf{c}}) \cup (Y \cap X^{\mathsf{c}}) \in I.$$

Let $[X] = \{\, Y \in \mathscr{L}(T) \mid X \sim Y\,\}$ denote the equivalence class of X. Then

$$\mathcal{T} := \{\, [X] \mid X \in \mathscr{L}(T)\,\}$$

gives rise to the quotient algebra $\langle \mathcal{T}, \sqcup, \sqcap, {}^{\mathsf{c}}, 0_{\mathcal{T}}, 1_{\mathcal{T}}\rangle$. This is a σ-complete Boolean algebra. For $X, Y \in \mathscr{L}(T)$, we have $[X] \sqcup [Y] = [X \cup Y]$, $[X] \sqcap [Y] = [X \cap Y]$, $[X]^{\mathsf{c}} = [X^{\mathsf{c}}]$, $0_{\mathcal{T}} = [\varnothing]$ and $1_{\mathcal{T}} = [T]$. For $s, t \in \mathcal{T}$, we define $s \sqsubseteq t$ iff $s \sqcup t = t$. If $X \subseteq Y$, then $[X] \sqsubseteq [Y]$. Whenever $X, Y \in t \in \mathcal{T}$, we have $\mu_{\mathrm{L}}(X) = \mu_{\mathrm{L}}(Y)$, so we can let $\mu_{\mathrm{L}}(t)$ be this unique value that μ_{L} assumes at any element of t. This way, μ_{L} can be extended to $\mathcal{T}$, and it is known that μ_{L} gives a positive measure on $\mathcal{T}$. Thus $\mathcal{T}$ is a measure algebra.

We can now forsake (4) and put forward the following definition:

Definition. For all $x \in D$:

- x **exists at** $t \in \mathcal{T}$ **with respect to** $\{\mu^p\}_{p\in T}$ iff $[\heartsuit_\mu(x)] \sqsupseteq t \neq 0_{\mathcal{T}}$.

- x **never exists with respect to** $\{\mu^p\}_{p\in T}$ iff $[\heartsuit_\mu(x)] = 0_{\mathcal{T}}$.

According to this definition, if x exists at $t \neq 0_{\mathcal{T}}$, then $\mu_{\rm L}([\heartsuit_\mu(x)]) \geq \mu_{\rm L}(t) > 0$ by Lemma 1 and the positivity of $\mu_{\rm L}$ on $\mathcal{T}$, so x's life has some positive length. On the other hand, if $\heartsuit_\mu(x) = \{p_0\}$, then $[\heartsuit_\mu(x)] = 0_{\mathcal{T}}$, so x never exists.[3]

Some immediate worries present themselves. In this new setting, a "time" is a member of $\mathcal{T}$, i.e., an equivalence class of Lebesgue-measurable subsets of T.[4] This means that if $X \subseteq T$ is not Lebesgue-measurable, we cannot even discuss whether an entity exists at X. Also, if $\heartsuit_\mu(x)$ is not Lebesgue-measurable, we cannot discuss whether x exists at any time. Indeed, using the axiom of choice, one can construct subsets of $\mathbb{R}$ that are not Lebesgue-measurable (Vitali, 1905). On the other hand, Zermelo–Fraenkel set theory without the axiom of choice has a model where every subset of $\mathbb{R}$ is Lebesgue-measurable (Solovay, 1970). What I take this to imply is that we probably need not worry about the possible existence of sets that are not Lebesgue-measurable. Even if they exist, since we obviously cannot mentally execute an infinite-step process involved in the use of the axiom of choice, I believe it is safe to assume that such sets do not figure in our mental model for natural language semantics.

Another worry concerns equating subsets of T whose symmetric differences are of measure zero. For instance, suppose that $\heartsuit_\mu(x) = (0, 1)\cup(1, 2)$.[5] Since $[\heartsuit_\mu(x)] = [(0, 2)]$, it follows that x exists at $[(0, 2)]$ by our definition. Such a statement, however, would naturally lead one to expect every nonzero part of x to measure positively throughout $(0, 2)$. Nevertheless, since $1 \notin \heartsuit_\mu(x)$, there is some nonzero part y of x such that $\mu^1(y) = 0$. This is quite counterintuitive. How can something only momentarily vanish and then come back? Makoto Kanazawa (personal communication) suggests that this should not be a problem from a measure theoretic point of view, since the probability of picking such a point is zero. More precisely, if $X \subseteq T$ and $X \sim \heartsuit_\mu(x)$, then $\mu_{\rm L}(X\cap\heartsuit_\mu(x)^{\rm c}) = 0$, so assuming that $\mu_{\rm L}(X) = \mu_{\rm L}(\heartsuit_\mu(x)) > 0$, the probability of picking a point p from X that is not in $\heartsuit_\mu(x)$ is

$$\frac{\mu_{\rm L}(X \cap \heartsuit_\mu(x)^{\rm c})}{\mu_{\rm L}(X)} = 0.$$

In measure theory, a property is said to hold **almost everywhere** if it does except on a set of measure zero. Analogously, in probability theory, an event is said to happen **almost surely** if its probability is 1. Let's shift to this way of thinking and speak as follows:

(5) $p \in X \sim \heartsuit_\mu(x)$ **almost surely entails** that $\mu^p(x) > 0$.

Thus, $[\heartsuit_\mu(x)] = [(0, 2)]$ almost surely entails that $\mu^1(x) > 0$.

Lastly, note that since $\heartsuit_\mu(0_D) = T$, it follows that 0_D exists at any $t \neq 0_{\mathcal{T}}$. I take this to be a mere technicality of no real semantic import.

4 Mass Noun Denotations

Let M be a mass noun. To describe situations where an entity becomes M or ceases to be M, the meaning of M needs to be sensitive to the time. So let's take M to denote a binary relation between times and entities. Now that our slogan "existence as positive measurement" motivates the view that the times that matter in human minds are members of $\mathcal{T}$, this means that M denotes a relation between the two Boolean algebras $\mathcal{T}$ and D, i.e., $[\![\mathrm{M}]\!] \subseteq \mathcal{T}\times D$. Let $[\![\mathrm{M}]\!]\,(t)$ denote the set of entities that are M at a given time t:

$$[\![\mathrm{M}]\!]\,(t) := \{\, x \in D \mid \langle t, x\rangle \in [\![\mathrm{M}]\!] \,\}.$$

As mentioned in Section 1, $[\![\mathrm{M}]\!]\,(t)$ ought to be a principal ideal. This means that there is a family $\{m_t\}_{t\in\mathcal{T}}$ of elements of D such that

$$[\![\mathrm{M}]\!]\,(t) = {\downarrow} m_t.$$

Now, let's consider the set of times at which a given entity x is M, for which we write:

$$[\![\mathrm{M}]\!]^{-1}(x) := \{\, t \in \mathcal{T} \mid \langle t, x\rangle \in [\![\mathrm{M}]\!] \,\}.$$

Intuitively, if x is M both at s and at t, then x ought to be M at $s \sqcup t$. Also, if x is M at t and $s \sqsubseteq t$, then x must be M at s as well. This means that $[\![\mathrm{M}]\!]^{-1}(x)$ should be an ideal in $\mathcal{T}$. Regarding this matter, the following holds.

[3] If $t = 0_{\mathcal{T}}$, then $[\heartsuit_\mu(x)] \sqsupseteq t$ trivially holds, so this case has been excluded from the definition in order to avoid such a clashing statement as "x exists at t and x never exists."

[4] In natural language semantics, "times" often mean time intervals. Aside from dealing with equivalence classes, our "times" are different from time intervals in that they can be discontinuous.

[5] (a, b) denotes an open interval. Thus $(0, 1) = \{\, p \in T \mid 0 < p < 1 \,\}$.

Proposition 3. *Suppose that each $\llbracket \mathrm{M} \rrbracket\,(t)$ is a principal ideal $\downarrow m_t$. Then, the following are equivalent:*

(a) $\llbracket \mathrm{M} \rrbracket^{-1}(x)$ *is a σ-complete ideal in $\mathcal{T}$ for all $x \in D$.*

(b) $\bigwedge_{t\in\mathcal{C}} m_t = m_{\bigsqcup \mathcal{C}}$ *for all countable $\mathcal{C} \subseteq \mathcal{T}$.*

Proof of this proposition follows shortly. Before that, note that what Condition (b) says is twofold. First, if x is M at s and y is M at t, then $x \wedge y$ is M at $s \sqcup t$. This is expected of M's denotation; since $x \wedge y$ is part both of x and of y, it ought to be M both at s and at t, and hence throughout $s \sqcup t$. Second, if something is M at $s \sqcup t$, then it must be M both at s and at t. Again, this is only expected.

Lemma 4. *Suppose that Condition (b) of Proposition 3 holds. Then, the following hold:*

(i) $m_{0_\mathcal{T}} = 1_D$.

(ii) *If $s \sqsubseteq t$, then $m_t \leq m_s$.*

Proof. (i) $m_{0_\mathcal{T}} = m_{\bigsqcup \varnothing} = \bigwedge_{t\in\varnothing} m_t = \bigwedge \varnothing = 1_D$.

(ii) If $s \sqsubseteq t$, then $t = s \sqcup t$, so $m_t = m_{s\sqcup t} = m_s \wedge m_t \leq m_s$. □

Proof of Proposition 3. (a) ⇒ (b). Let $\mathcal{C} \subseteq \mathcal{T}$ be countable.

Suppose that $x \leq m_t$ for all $t \in \mathcal{C}$. Then for all $t \in \mathcal{C}$, we have $x \in \downarrow m_t = \llbracket \mathrm{M} \rrbracket\,(t)$, so $\langle t, x\rangle \in \llbracket \mathrm{M} \rrbracket$ and hence $t \in \llbracket \mathrm{M} \rrbracket^{-1}(x)$. Since $\llbracket \mathrm{M} \rrbracket^{-1}(x)$ is a σ-complete ideal by assumption, it follows that $\bigsqcup \mathcal{C} \in \llbracket \mathrm{M} \rrbracket^{-1}(x)$, so $x \in \llbracket \mathrm{M} \rrbracket\,(\bigsqcup \mathcal{C}) = \downarrow m_{\bigsqcup \mathcal{C}}$ and hence $x \leq m_{\bigsqcup \mathcal{C}}$. Taking $\bigwedge_{t\in\mathcal{C}} m_t$ for x in particular, we obtain $\bigwedge_{t\in\mathcal{C}} m_t \leq m_{\bigsqcup \mathcal{C}}$.

Next, suppose that $x \leq m_{\bigsqcup \mathcal{C}}$. Then $x \in \downarrow m_{\bigsqcup \mathcal{C}} = \llbracket \mathrm{M} \rrbracket\,(\bigsqcup \mathcal{C})$, so $\bigsqcup \mathcal{C} \in \llbracket \mathrm{M} \rrbracket^{-1}(x)$. For every $t \in \mathcal{C}$, we have $t \leq \bigsqcup \mathcal{C}$, and since $\llbracket \mathrm{M} \rrbracket^{-1}(x)$ is a σ-complete ideal by assumption, it follows that $t \in \llbracket \mathrm{M} \rrbracket^{-1}(x)$, and hence $x \in \llbracket \mathrm{M} \rrbracket\,(t) = \downarrow m_t$, so $x \leq m_t$. It follows that $x \leq \bigwedge_{t\in\mathcal{C}} m_t$. Taking $m_{\bigsqcup \mathcal{C}}$ for x in particular, we obtain $m_{\bigsqcup \mathcal{C}} \leq \bigwedge_{t\in\mathcal{C}} m_t$.

This completes the proof that $\bigwedge_{t\in\mathcal{C}} m_t = m_{\bigsqcup \mathcal{C}}$.

(b) ⇒ (a). We verity that $\llbracket \mathrm{M} \rrbracket^{-1}(x)$ satisfies the three conditions for being a σ-complete ideal.

(C1) $x \leq 1_D = m_{0_\mathcal{T}}$ by Lemma 4(i), so $x \in \downarrow m_{0_\mathcal{T}} = \llbracket \mathrm{M} \rrbracket\,(0_\mathcal{T})$. Thus, $\langle 0_\mathcal{T}, x\rangle \in \llbracket \mathrm{M} \rrbracket$ and $0_\mathcal{T} \in \llbracket \mathrm{M} \rrbracket^{-1}(x)$.

(C2′) Suppose that $\mathcal{C} \subseteq \mathcal{T}$ is countable, and $t \in \llbracket \mathrm{M} \rrbracket^{-1}(x)$ for all $t \in \mathcal{C}$. Then for all $t \in \mathcal{C}$, we have $x \in \llbracket \mathrm{M} \rrbracket\,(t) = \downarrow m_t$ and hence $x \leq m_t$, so $x \leq \bigwedge_{t\in\mathcal{C}} m_t = m_{\bigsqcup \mathcal{C}}$ by the assumed condition. Thus, $x \in \downarrow m_{\bigsqcup \mathcal{C}} = \llbracket \mathrm{M} \rrbracket\,(\bigsqcup \mathcal{C})$ and hence $\bigsqcup \mathcal{C} \in \llbracket \mathrm{M} \rrbracket^{-1}(x)$.

(C3) Suppose that $t \in \llbracket \mathrm{M} \rrbracket^{-1}(x)$ and $s \sqsubseteq t$. Since $x \in \llbracket \mathrm{M} \rrbracket\,(t) = \downarrow m_t$, we have $x \leq m_t \leq m_s$ by Lemma 4(ii), so $x \in \downarrow m_s = \llbracket \mathrm{M} \rrbracket\,(s)$. Hence $s \in \llbracket \mathrm{M} \rrbracket^{-1}(x)$. □

I think it is reasonable to assume that $\llbracket \mathrm{M} \rrbracket^{-1}(x)$ is indeed a σ-complete ideal in $\mathcal{T}$, and a principal ideal at that, just as $\llbracket \mathrm{M} \rrbracket\,(t)$ is a principal ideal in D. This means that there is a family $\{\ell_x\}_{x\in D}$ of elements of $\mathcal{T}$ such that

$$\llbracket \mathrm{M} \rrbracket^{-1}(x) = \downarrow \ell_x.^{6}$$

As times and entities play symmetric roles in M's denotation, the following are immediate as parallels of Proposition 3 and Lemma 4.

Proposition 5. *Suppose that each $\llbracket \mathrm{M} \rrbracket^{-1}(x)$ is a principal ideal $\downarrow \ell_t$. Then, the following are equivalent:*

(a) $\llbracket \mathrm{M} \rrbracket\,(t)$ *is a σ-complete ideal in D for all $t \in \mathcal{T}$.*

(b) $\bigsqcap_{x\in C} \ell_x = \ell_{\bigvee C}$ *for all countable $C \subseteq D$.*

Lemma 6. *Suppose that Condition (b) of Proposition 5 holds. Then, the following hold:*

(i) $\ell_{0_D} = 1_\mathcal{T}$.

(ii) *If $x \leq y$, then $\ell_y \sqsubseteq \ell_x$.*

What is the meaning of Condition (b) of Proposition 5? First, it says that if x is M at s and y is M at t, then $x \vee y$ is M at $s \sqcap t$. Second, it says that if $x \vee y$ is M at some time, then both x and y are M at that time.

Let $\mathcal{M} = \{\, m_t \mid t \in \mathcal{T} \,\}$ and $\mathcal{L} = \{\, \ell_x \mid x \in D \,\}$. Through domain restriction, ℓ can be regarded as a function from $\mathcal{M}$ into $\mathcal{L}$. Likewise, m can be viewed as a function from $\mathcal{L}$ into $\mathcal{M}$. Then the pair of ℓ and m forms what is known as a **Galois connection** between $\langle \mathcal{M}, \leq\rangle$ and $\langle \mathcal{L}, \sqsubseteq\rangle$, satisfying the condition in (i) below (*vide* Mac Lane, 1998).

Lemma 7. *Suppose that $x \in D$ and $t \in \mathcal{T}$.*

(i) *$t \sqsubseteq \ell_x$ if and only if $x \leq m_t$.*

(ii) *$x \leq m_{\ell_x}$ and $t \sqsubseteq \ell_{m_t}$.*

[6] Speaking intuitively, ℓ_x is x's lifespan as M.

(iii) $m_{\ell_{m_t}} = m_t$ *and* $\ell_{m_{\ell_x}} = \ell_x$.

Proof. (i) $t \sqsubseteq \ell_x$ iff $t \in \downarrow\ell_x$ iff $t \in [\![\mathrm{M}]\!]^{-1}(x)$ iff $\langle t, x\rangle \in [\![\mathrm{M}]\!]$ iff $x \in [\![\mathrm{M}]\!]\,(t)$ iff $x \in \downarrow m_t$ iff $x \le m_t$.

(ii) Since $\ell_x \in \downarrow\ell_x = [\![\mathrm{M}]\!]^{-1}(x)$, we have $\langle \ell_x, x\rangle \in [\![\mathrm{M}]\!]$, so $x \in [\![\mathrm{M}]\!]\,(\ell_x) = \downarrow m_{\ell_x}$ and hence $x \le m_{\ell_x}$. Dually for $t \sqsubseteq \ell_{m_t}$.

(iii) By (ii), $m_t \le m_{\ell_{m_t}}$. Also, since $t \sqsubseteq \ell_{m_t}$ by (ii), we have $m_{\ell_{m_t}} \le m_t$ by Lemma 4(ii). Hence $m_{\ell_{m_t}} = m_t$. Dually for $\ell_{m_{\ell_x}} = \ell_x$. □

Our denotation of a mass noun M is a binary relation between equivalence classes of Lebesgue-measurable sets of time points and entities. However, we should also like to be able to talk about whether or not an entity is M at a specific time point. I say that a natural idea would be that an entity is M at a time point p if and only if it is M at some open set (in topological sense) of time points that contains p. Let's define the derivative denotation of M for a time point $p \in T$ by

Definition. $$[\![\mathrm{M}]\!]\,(p) := \bigcup_{G\ \text{open},\ p\in G} [\![\mathrm{M}]\!]\,([G]).$$

Note that every open set G is Lebesgue-measurable, so $[G]$ makes sense. Here is a reason for considering only open sets. Imagine that for an entity x, we have $\ell_x = [(0,1)] = [[0,1]]$.[7] This means that x begins to be M at time point 0 and ceases to be M at time point 1. Now, do we want to say that x is M at time point 0? How about at time point 1? Some might say yes, but it seems fair to say that it is uncertain whether x is M at those points. If we allowed G to be the non-open set $[0,1]$ in the definition of $[\![\mathrm{M}]\!]\,(0)$, we would be forced to say that x is M at 0. By dealing exclusively with open sets, we can avoid such an undesirable conclusion. On our definition, in contrast, if $x \in [\![\mathrm{M}]\!]\,(p)$, then there is an open set G such that $p \in G$ and $x \in [\![\mathrm{M}]\!]([G])$. As G is open, there is an open neighborhood of p in G, or more specifically, an open interval B such that $p \in B \subseteq G$. Since $[G] \in [\![\mathrm{M}]\!]^{-1}(x)$ and $[\![\mathrm{M}]\!]^{-1}(x)$ is an ideal, it follows that $[B] \in [\![\mathrm{M}]\!]^{-1}(x)$. Thus, we conclude that x is M at some time interval that surrounds p instead of having p on its edge. Assuming that $[\![\mathrm{M}]\!]\,(t)$ and $[\![\mathrm{M}]\!]^{-1}(x)$ are principal ideals, our definition leads to the following nice consequence.

[7] $[0,1]$ denotes the closed interval $\{\, p \in T \mid 0 \le p \le 1 \,\}$, and $[[0,1]]$ its equivalence class.

Proposition 8. $[\![\mathrm{M}]\!]\,(p)$ *is an ideal in* D.

Proof. We check the three conditions for being an ideal.

(C1) Since $0_D \le m_{1_\mathcal{T}}$, we have $0_D \in \downarrow m_{1_\mathcal{T}} = [\![\mathrm{M}]\!]\,(1_\mathcal{T}) = [\![\mathrm{M}]\!]\,([T])$. Since the whole space T is open and $p \in T$, this shows that $0_D \in [\![\mathrm{M}]\!]\,(p)$.

(C2) Suppose that $x, y \in [\![\mathrm{M}]\!](p)$. Then, there exist an open set G_1 such that $p \in G_1$ and $x \in [\![\mathrm{M}]\!]\,([G_1])$, and an open set G_2 such that $p \in G_2$ and $y \in [\![\mathrm{M}]\!]\,([G_2])$. Then $[G_1] \in [\![\mathrm{M}]\!]^{-1}(x) = \downarrow\ell_x$, so $[G_1] \sqsubseteq \ell_x$. Similarly, $[G_2] \sqsubseteq \ell_y$. Then $[G_1 \cap G_2] = [G_1] \sqcap [G_2] \sqsubseteq \ell_x \sqcap \ell_y = \ell_{x\vee y}$ by Proposition 5(b). So $[G_1 \cap G_2] \in \downarrow\ell_{x\vee y} = [\![\mathrm{M}]\!]^{-1}(x \vee y)$ and hence $x \vee y \in [\![\mathrm{M}]\!]\,([G_1 \cap G_2])$. As $G_1 \cap G_2$ is an open set and $p \in G_1 \cap G_2$, this shows that $x \vee y \in [\![\mathrm{M}]\!]\,(p)$.

(C3) Suppose that $x \in [\![\mathrm{M}]\!](p)$ and $y \le x$. Then, there is an open set G such that $p \in G$ and $x \in [\![\mathrm{M}]\!]([G]) = \downarrow m_{[G]}$. Then $y \le x \le m_{[G]}$, so $y \in \downarrow m_{[G]} = [\![\mathrm{M}]\!]([G])$. Hence $y \in [\![\mathrm{M}]\!](p)$. □

Note that $[\![\mathrm{M}]\!](p)$ is not necessarily a σ-complete ideal. This is due to the fact that countable intersection of open sets need not be an open set. For instance, $G_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ is open for every positive integer n, but $\bigcap_{n\ge1} G_n = \{0\}$ is not.

We have derived $[\![\mathrm{M}]\!](p)$'s from $[\![\mathrm{M}]\!]([G])$'s, but we can also get back to $[\![\mathrm{M}]\!]([G])$'s from $[\![\mathrm{M}]\!](p)$'s.

Proposition 9. *Let* $G \subseteq T$ *be an open set. Then* $[\![\mathrm{M}]\!]\,([G]) = \bigcap_{p\in G} [\![\mathrm{M}]\!]\,(p)$.

Proof. By the definition of $[\![\mathrm{M}]\!]\,(p)$, for every $p \in G$, we have $[\![\mathrm{M}]\!]\,([G]) \subseteq [\![\mathrm{M}]\!]\,(p)$, so $[\![\mathrm{M}]\!]\,([G]) \subseteq \bigcap_{p\in G} [\![\mathrm{M}]\!]\,(p)$.

For the reverse inclusion, suppose that $x \in \bigcap_{p\in G} [\![\mathrm{M}]\!]\,(p)$. For each $p \in G$, we have $x \in [\![\mathrm{M}]\!]\,(p)$, so there is some open set G_p such that $p \in G_p$ and $x \in [\![\mathrm{M}]\!]\,([G_p]) = \downarrow m_{[G_p]}$ and hence $x \le m_{[G_p]}$. Since $G \subseteq \bigcup_{p\in G} G_p$, $\{G_p\}_{p\in G}$ is an open cover of G. Since $T = \mathbb{R}$ has a countable base for its topology, there exists a countable subcover $\{G_{p_n}\}_{n\in\mathbb{N}}$, so $G \subseteq \bigcup_{n\in\mathbb{N}} G_{p_n}$. Then $[G] \sqsubseteq \left[\bigcup_{n\in\mathbb{N}} G_{p_n}\right]$, and

$$\begin{aligned} x &\le \bigwedge_{n\in\mathbb{N}} m_{[G_{p_n}]} && (\text{since } x \le m_{[G_{p_n}]} \text{ for all } n)\\ &= m_{\bigsqcup_{n\in\mathbb{N}}[G_{p_n}]} && (\text{by Proposition 3(b)})\\ &= m_{[\bigcup_{n\in\mathbb{N}} G_{p_n}]}\\ &\le m_{[G]}, && (\text{by Lemma 4(ii)}) \end{aligned}$$

so $x \in \downarrow m_{[G]} = [\![\mathrm{M}]\!]\,([G])$. This shows that $\bigcap_{p\in G} [\![\mathrm{M}]\!]\,(p) \subseteq [\![\mathrm{M}]\!]\,([G])$. □

It is time we revisited mass entities' lives. If something is M at some time, then that thing had better exist at that time in the sense defined in the previous section. I would now like to propose the following:

(6) For every mass noun M, there is a relevant family $\{\mu^p\}_{p \in T}$ of measures such that $t \sqsubseteq [\heartsuit_\mu(x)]$ for all $t \in \mathcal{T}$ and $x \in D$ such that $\langle t, x \rangle \in [\![\text{M}]\!]$.

This does not mean that there is a unique relevant measure for a given mass noun. For instance, milk can be measured in terms of volume (*liters of*) or in terms of mass (*grams of*). (6) ensures that even if sentences do not explicitly use expressions like *liters of*, some implicit measure is involved in modeling their meaning. In fact, the sentences in (7) contain no specific measuring unit, and yet demonstrate that the amount of milk is somehow measured; otherwise, one could not state if the milk in the tank is a lot or a little. As (8) shows, the same point holds for nouns like *love* as well, for which no appropriate word of a measuring unit seems to exist.

(7) a. There is a lot of milk in the tank.
b. There is a little milk in the tank.

(8) a. There is a lot of love between Christa and Ymir.
b. There is a little love between Christa and Ymir.

Why shouldn't we rather say the following stronger statement instead of (6)?

(6′) For every mass noun M, there is a relevant family $\{\mu^p\}_{p \in T}$ of measures such that $\ell_x = [\heartsuit_\mu(x)]$ for all $x \in D$.

This says that the maximum time at which x is, say, milk coincides with x's life, i.e., the time throughout which x has some physical presence. However, we do not want to stipulate that x be allowed to exist only as milk. Imagine that x starts its life as milk at time point 0, turns into cheese at time point 1, and eventually gets eaten by Chris at time point 2 to completely vanish from the world. In this case, $\ell_x = [(0, 1)]$ and $[\heartsuit_\mu(x)] = [(0, 2)]$, so $\ell_x \neq [\heartsuit_\mu(x)]$. The following proposition shows that (6) can be equivalently expressed in different ways.

Proposition 10. *The following are equivalent:*

(i) $t \sqsubseteq [\heartsuit_\mu(x)]$ *for all* $t \in \mathcal{T}$ *and* $x \in D$ *such that* $\langle t, x \rangle \in [\![\text{M}]\!]$.

(ii) $\ell_x \sqsubseteq [\heartsuit_\mu(x)]$ *for all* $x \in D$.

(iii) $t \sqsubseteq [\heartsuit_\mu(m_t)]$ *for all* $t \in \mathcal{T}$.

Proof. (i) ⇒ (ii). Let $x \in D$. Since $\ell_x \in \downarrow\!\ell_x = [\![\text{M}]\!]^{-1}(x)$, we have $\langle \ell_x, x \rangle \in [\![\text{M}]\!]$. The assumed condition (i) then implies that $\ell_x \sqsubseteq [\heartsuit_\mu(x)]$.

(ii) ⇒ (iii). Let $t \in \mathcal{T}$. By Lemma 7(ii) and the assumed condition (ii), $t \sqsubseteq \ell_{m_t} \sqsubseteq [\heartsuit_\mu(m_t)]$.

(iii) ⇒ (i). Suppose that $\langle t, x \rangle \in [\![\text{M}]\!]$. Since $t \in [\![\text{M}]\!]^{-1}(x) = \downarrow\!\ell_x$, we have $t \sqsubseteq \ell_x$. By the assumed condition (iii), $\ell_x \sqsubseteq [\heartsuit_\mu(m_{\ell_x})]$. Since $x \leq m_{\ell_x}$ by Lemma 7(ii), $\heartsuit_\mu(m_{\ell_x}) \subseteq \heartsuit_\mu(x)$ by Lemma 2(ii). Hence $t \sqsubseteq \ell_x \sqsubseteq [\heartsuit_\mu(m_{\ell_x})] \sqsubseteq [\heartsuit_\mu(x)]$. □

Finally, let's get back to the sentences in (2). They can now be analyzed as follows, where p^* denotes the utterance time point:

(9) a. There is some x such that $x \in [\![\text{milk in the tank}]\!](p^*)$.
b. There is some x such that $x \in [\![\text{milk in the tank}]\!](p^*)$ and $\mu^{p^*}_{\text{liter}}(x) \geq 9.8$.

Note that a number of assumptions are at work under cover of the deceptively simple appearance of (9-a). By the definition of the derivative denotation for time points, (9-a) means that there exists an open $G \subseteq T$ such that $p^* \in G$ and $x \in [\![\text{milk in the tank}]\!]([G])$. By (6), there is a relevant family $\{\mu^p\}_{p \in T}$ of measures and $[G] \sqsubseteq [\heartsuit_\mu(x)]$. Adopting the way of talking in (5), this almost surely entails that $\mu^{p^*}(x) > 0$.

5 Continuous Production/Consumption

Let's move on to sentences describing continuous production or consumption of mass entities like the following:

(10) a. The cow produced 9.8 liters of milk yesterday.
b. The calf consumed 9.8 liters of milk yesterday.

When the subject is fixed to some particular individual like *the cow* or *the calf*, transitive verbs may be regarded as denoting a binary relation between times and entities.

While not necessary for analyzing (10), it might be illuminating to compare the denotations of these verbs with mass noun denotations.

An important property of these verbs is what might be called double cumulativity,[8] in the sense illustrated below:

(11) the cow produced x at s and y at t
$\Longrightarrow$ the cow produced $x \vee y$ at $s \sqcup t$

This property implies that the verb denotation is cumulative separately in the entity domain and in the temporal domain:[9]

(12) a. the cow produced x at t and y at t
$\Longrightarrow$ the cow produced $x \vee y$ at t
b. the cow produced x at s and x at t
$\Longrightarrow$ the cow produced x and $s \sqcup t$

However, these separate cumulative properties in the two domains do not entail double cumulativity. Indeed, the denotation of a mass noun is cumulative in the two separate domains as detailed in the previous section, but is not doubly cumulative:

(13) x is milk at s and y is milk at t
$\not\Longrightarrow$ $x \vee y$ is milk at $s \sqcup t$

Also, I say that verbs V of production or consumption (with a fixed singular subject) have double distributivity[10] in the following sense:

(14) If $\langle t, x\rangle \in [\![\mathrm{V}]\!]$, then for all nonzero $s \sqsubseteq t$, there is some nonzero $y \leq x$ such that $\langle s, y\rangle \in [\![\mathrm{V}]\!]$, and for all nonzero $y \leq x$, there is some nonzero $s \sqsubseteq t$ such that $\langle s, y\rangle \in [\![\mathrm{V}]\!]$.

However, this does not imply distributivity in each separate domain:

(15) a. the cow produced x at t and $y \leq x$
$\not\Longrightarrow$ the cow produced y at t
b. the cow produced x at t and $s \sqsubseteq t$
$\not\Longrightarrow$ the cow produced x at s

To understand this, imagine that the cow spent the time period $[(0,1)]$ to produce y, a total of 5.5 liters of milk, and spent the time period $t = [(0,2)]$ to produce x, a total of 9.8 liters of milk. Let $s = [(1,2)]$. Then $y \leq x$ and $s \sqsubseteq t$, but no nonzero part of y was produced at any nonzero part of s. The assumed double distributivity then entails that the cow did not produce y at t nor did it produce x at s. This concludes that unlike in the case of mass nouns, neither $[\![\mathrm{V}]\!]\,(t)$ nor $[\![\mathrm{V}]\!]^{-1}(x)$ is an ideal. I would like to note, however, that the weaker form of distributivity in (16-a) should hold, as evidenced by the valid inference in (16-b).

(16) a. the cow produced $x \neq 0_D$ at t
$\Longrightarrow$ the cow produced some nonzero proper part of x at t
b. the cow produced 9.8 liters of milk at t
$\Longrightarrow$ the cow produced 5.5 liters of milk at t

Now, let me hope that I can come back to study lexical denotations of verbs of production/consumption in more detail in the future, and let me return to (10). A straightforward analysis of (10-a) will look something like the following:

(17) There is some x such that $x \in [\![\mathrm{milk}]\!](p^*)$, $\mu^{p^*}_{\mathsf{liter}}(x) \geq 9.8$, and $\langle \mathsf{ystd}, x\rangle \in [\![\text{the cow produced}]\!]$,

where p^* denotes the utterance time point and $\mathsf{ystd} \in \mathcal{T}$ is the equivalence class of the whole interval of yesterday. (17) can be equivalently expressed as follows:

$$\mu^{p^*}_{\mathsf{liter}}\Big(\bigvee\big([\![\mathrm{milk}]\!]\,(p^*) \cap [\![\text{the cow produced}]\!]\,(\mathsf{ystd})\big)\Big) \geq 9.8.$$

Imagine, however, that the calf consumed all the milk produced by the cow within yesterday. Then no portion of the milk in question remains today, so (17) will come out false. Can we perhaps fix this problem by changing p^* to some appropriate time point in yesterday? No. Imagine that the calf drank the milk directly from the breast of the dam as she produced it. If the calf's swallowing milk is to be understood as disappearance of the milk from the world, the milk in question that existed at any given time point p is the milk that was in the mouth of the calf at p, and it is quite possible that its volume was always below 9.8 liters.[11] We then might want to split yesterday into small subintervals, and sum up the amounts of milk that the cow produced at those

[8] Krifka (1989) calls this property *Summativity* for relations between events and objects.

[9] Although the kind of situation described in (10-b) is unrealistic, I see nothing wrong with the inference *per se*; so long as one can imagine one and the same entity can be produced twice, one sees that the inference goes through.

[10] This corresponds to the properties Krifka (1989) calls *Mapping to Objects* and *Mapping to Events*.

[11] Szabó (2006) discusses a similar problem involving count nouns such as *Helen had three husbands*, which can be true even if there was no past time when Helen had three husbands simultaneously.

intervals. I argue, however, that that appraoch is still insufficient for two reasons.

To illustrate my first point, let's say that within yesterday, the cow started milk production at time point p_0 and ended it at p_n. Then, we need only look at (the equivalence class of) the interval $[(p_0, p_n)]$. Let's say that we split this into n subintervals $t_1 = [(p_0, p_1)]$, $t_2 = [(p_1, p_2)]$, ..., $t_n = [(p_{n-1}, p_n)]$ and want to see whether something like the following holds:

(18) $$\sum_{k=1}^{n} \mu^{p_k}_{\text{liter}} \Big(\bigvee ([\![\text{milk}]\!] \, (p_k) \cap [\![\text{the cow produced}]\!] \, (t_k)) \Big) \geq 9.8.$$

Unfortunately, exactly the same problem persists for each subinterval in principle: there is no guarantee that the portion of the milk produced during t_k remained wholly at p_k. However, by making the splitting finer and finer, we can expect to obtain a more and more precise measurement of the total milk production.

The second reason has to do with the fact that even in a situation where an entity keeps existing, its measurement can change throughout its life. (10-a) can be naturally used when the hearer has no idea that such milk existed. In that case, *9.8 liters of milk* is **nonpresuppositional**, and we will focus on this reading in this paper. Musan (1995) observes that the temporal interpretation of a nonpresuppositional noun phrase is obligatorily temporally dependent on the main predicate. In the German example in (19), unless *Einige* is stressed, which would indicate presuppositionality, the temporal interpretation of *Einige Professoren* must coincide with that of *glücklich*, so the sentence is only understood as talking about individuals who were professors in the sixties.

(19) Einige Professoren waren in den sechziger Jahren glücklich.
some professors were in the sixties happy
(Musan, 1995, p. 79)

As for (10-a), we see that it talks about entities that were milk at the time of production, as Musan's generalization predicts. What about the temporal interpretation of the measuring phrase *9.8 liters*? Intuitively, what matters here is the volume of milk as measured at the time of production. As a matter of fact, milk shrinks in volume when cooled. So if the produced milk was refrigerated in the tank after the milking, then its volume must have become smaller than at the time of production. However, none of such concerns seem to matter in judging whether (10-a) is true.[12] I therefore suppose that Musan's generalization extends to measuring phrases (Shimada, 2009). That is to say, the temporal interpretation of the measuring phrase of a nonpresuppositional noun phrase is obligatorily dependent on the main predicate. But then, we see a problem in the summands in (18), which are:

$$\mu^{p_k}_{\text{liter}} \Big(\bigvee ([\![\text{milk}]\!] \, (p_k) \cap [\![\text{the cow produced}]\!] \, (t_k)) \Big).$$

Here, what is measured is the milk produced during t_k, but it is measure at p_k, the endpoint of t_k. This means that what is produced, say, in the first half of t_k is not measured at the time of its production. However, if we assume that volume change can only be so gradual, then making the splitting finer and finer will lead to a more and more precise measurement.

At this point, it must be evident that we need to look at infinitesimally small subintervals. In other words, we need the full power of integration. Indeed, Krifka (1989) suggests using a semantically well-motivated "calculus" (p. 91) to analyze such expressions as *drink wine*, and my idea is to carry this out literally.[13] Now, the set of entities that the cow produced during $[(p, p + h)]$, where $h > 0$, is given by

$$[\![\text{the cow produced}]\!] \, ([(p, p + h)]).$$

Suppose that x belongs to this set. Is x milk at $[(p, p + h)]$? Definitely not. Halfway into this time period, say at $p + h/2$, some part of x has yet to be produced, so not only was that part not milk, but it had measure zero. Therefore, in order to correctly predicate the mass noun *milk* of x, it has to be done at $p + h$, when all of x has been produced (ignoring the possible partial loss of x due to the calf consuming it by that point). Then, the total milk that the cow produced during $[(p, p + h)]$, for

[12] Granted that in reality, the volume of milk does not change so wildly as its temperature changes, but if one imagines that it does, this intuition will be clearer.

[13] In Shimada (2009), I treated mass nouns and verbs of production/consumption as essentially denoting relations between time points and entities. I think that that was the wrong idea, and now I am a proponent of the view that "times" that nouns and verbs are predicated of have positive length. From a measure-theoretic perspective, points are as good as nothingness. Simply put, size matters!

which we shall write $\alpha_{\mathsf{p}}([(p, p+h)])$, is

$$\alpha_{\mathsf{p}}([(p, p+h)]) = \bigvee(\llbracket\text{milk}\rrbracket\,(p+h) \cap \llbracket\text{the cow produced}\rrbracket\,([(p, p+h)])),$$

and its size can be meaningfully measured, again, only at $p+h$, which is

$$\mu^{p+h}_{\mathsf{liter}}(\alpha_{\mathsf{p}}([(p, p+h)])).$$

Since the average rate of milk production during $[(p, p+h)]$ in terms of measurement at $p+h$ is given by dividing the above by h, the rate $\varrho_{\mathsf{p}}(p)$ of milk production at p will be obtained by taking its limit:

$$\varrho_{\mathsf{p}}(p) = \lim_{h\to 0^+} \frac{\mu^{p+h}_{\mathsf{liter}}(\alpha_{\mathsf{p}}([(p, p+h)]))}{h}.$$

The total amount of milk production is obtained by integrating it over yesterday, so the truth conditions of (10-a) will be expressed as

$$\int_{\mathsf{ystd}} \varrho_{\mathsf{p}}\, d\mu_{\mathrm{L}} \geq 9.8.$$

This is a **Lebesgue integral** (*vide* Halmos, 1950), and μ_{L} is the Lebesgue measure. Similarly, the rate $\varrho_{\mathsf{c}}(p)$ of milk consumption at p can be calculated by

$$\varrho_{\mathsf{c}}(p) = \lim_{h\to 0^+} \frac{\mu^{p-h}_{\mathsf{liter}}(\alpha_{\mathsf{c}}([(p-h, p)]))}{h},$$

where

$$\alpha_{\mathsf{c}}([(p-h, p)]) = \bigvee(\llbracket\text{milk}\rrbracket\,(p-h) \cap \llbracket\text{the calf consumed}\rrbracket\,([(p-h, p)])).$$

To calculate $\varrho_{\mathsf{c}}(p)$, we now have to look at intervals ending at p since the milk consumed by the calf no longer exists after p. (10-b) can then be analyzed as

(20) $$\int_{\mathsf{ystd}} \varrho_{\mathsf{c}}\, d\mu_{\mathrm{L}} \geq 9.8.$$

6 Telicity and Integration

Since Verkuyl (1972), it has been known that the type of direct object can affect the aspectual interpretation of the whole VP. If the direct object is a bare noun as in (21), the VP becomes **atelic** (i.e. having no endpoint) and can take a *for* temporal PP.

(21) The calf consumed milk for 90 seconds yesterday.

In contrast, if the direct object is quantified as in (22) and (23), the VP becomes **telic** (i.e. having an endpoint) and incompatible with a *for* temporal PP.

(22) a. The calf consumed 9.8 liters of milk yesterday. (= (10-b))
b. *The calf consumed 9.8 liters of milk for 90 seconds yesterday.

(23) a. The calf consumed some milk yesterday.
b. *The calf consumed some milk for 90 seconds yesterday.

In this final section, I will sketch how our approach might deal with this phenomenon, leaving a fuller theoretical development and comparisons with existing theories (Krifka, 1989, 1992, 1998; Zucchi and White, 2001; Rothstein, 2004; Kovalev, 2024 *inter alia*) for a future occasion.

First, we need to know how *for* temporal PPs are analyzed. Obviously, they measure the length of times, and I say that this is achieved by means of integration of functions from T into $\mathbb{R}$. For concreteness, let's consider the following sentence:

(24) Ymir ran for six hours yesterday.

Assuming that the verb *ran* denotes a relation between times and entities, the denotation of *Ymir ran* will be the set of times where Ymir ran.[14] To calculate the total length of Ymir's running, we need to get the set R of time points at which Ymir was running, and this may be given by the following (cf. our definition and discussion of the mass noun denotation for time points):

$$R = \bigcup\{\, G \mid G \text{ open}, G \neq \varnothing, [G] \in \llbracket\text{Ymir ran}\rrbracket \,\}.$$

The PP *for six hours* says that the intersection of this set and the set of time points in yesterday is at least $6 \cdot 60^2$ seconds long. Assuming that μ_{L} measures times in seconds, this can be expressed, using a Lebesgue integral, as

$$\int_{\mathsf{ystd}} \chi_R\, d\mu_{\mathrm{L}} \geq 6 \cdot 60^2,$$

where χ_R is the characteristic function of R.

[14] Authors like Taylor (1977) and Dowty (1979) argue that activities (such as *run*) have minimal parts, and if that is the case, $\llbracket$Ymir ran$\rrbracket$ will not be an ideal in $\mathcal{T}$.

Let's return to (21). We can regard *the calf consumed milk* as denoting the set of times where the calf consumed milk. If we are to trace our analysis of (24) above, we should derive from this the set of time points at which the calf was consuming milk, and integrate its characteristic function. However, in this case, we can directly get to the time points at which the calf was consuming milk, as they must coincide with the time points at which the rate of milk consumption is positive.[15] Using ϱ_{c} from the previous section, (21) will then be analyzed as follows:

(25) $$\int_{\mathsf{ystd}} \chi_{\{\, p \in T \mid \varrho_{\mathsf{c}}(p) > 0 \,\}} \, d\mu_{\mathrm{L}} \geq 90.$$

It should now be apparent why (22-b) does not work. Both the expressions *9.8 liters* and *for 90 seconds* require integration of a function from T into $\mathbb{R}$, so we need two integrals. However, once the time-point variable is "used up" by one integral, it will no longer be available for the other. This point might become transparent if we rewrite the integrands in (20) and (25) in λ notation:

(20′) $$\int_{\mathsf{ystd}} [\lambda p \in T.\, \varrho_{\mathsf{c}}(p)] \, d\mu_{\mathrm{L}} \geq 9.8.$$

(25′) $$\int_{\mathsf{ystd}} [\lambda p \in T.\, \varrho_{\mathsf{c}}(p) > 0] \, d\mu_{\mathrm{L}} \geq 90.$$

As you can see, the time-point variable p gets bound in an integral, so it cannot be further used to define a meaningful integrand for another integral. This situation is analogous to the variable binding in the formula $\forall x \exists x P(x)$; the inner quantifier $\exists x$ binds the occurrence of x in $P(x)$, and as a result, the outer quantifier $\forall x$ cannot bind it.

The contrast in (23) presents difficulties to theories like Krifka's, which will expect the VP to be atelic because proper parts of some milk are still some milk. Our approach treats (23) in much the same way as (22). (23-a) involves integration of the rate of milk-consumption just like (22-a), except that the consumed amount is unspecified, so it is analyzed as follows:

$$\int_{\mathsf{ystd}} \varrho_{\mathsf{c}} \, d\mu_{\mathrm{L}} > 0.$$

[15] Accordingly, $[\![$the calf consumed milk$]\!]$ as a set of times will be given as the following principal ideal in $\mathcal{T}$:

$$[\![\text{the calf consumed milk}]\!] = {\downarrow}[\{\, p \in T \mid \varrho_{\mathsf{c}}(p) > 0 \,\}].$$

(23-b) is bad for the same reason as (22-b) is.

Finally, we note that *in* temporal PPs will correspond to intervals of integration. It has been observed in the literature that *in* temporal PPs go with telic VPs, as demonstrated by (26).

(26) The calf consumed 9.8 liters of milk in two days.

Here, the PP *in two days* plays the same role as *yesterday* in (22-a), except that it is quantified. Therefore, it can be analyzed as something like the following:

(27) There is a time interval I (beginning at some contextually salient time point) such that $\mu_{\mathrm{L}}(I) = 2 \cdot 24 \cdot 60^2$ and

$$\int_I \varrho_{\mathsf{c}} \, d\mu_{\mathrm{L}} \geq 9.8.$$

In fact, a *for* temporal PP and an *in* temporal PP can co-occur as in (28), and this can be straightforwardly analyzed in much the same way, as shown in (29), where R is as defined earlier.

(28) Ymir ran for six hours in two days.

(29) There is a time interval I (beginning at some contextually salient time point) such that $\mu_{\mathrm{L}}(I) = 2 \cdot 24 \cdot 60^2$ and

$$\int_I \chi_R \, d\mu_{\mathrm{L}} \geq 6 \cdot 60^2.$$

7 Conclusion

By examining sentences involving mass nouns, we outlined an ontology based on the view that to be is to measure positively. Regarding mass noun denotations, we argued that our intuition is best captured if they are thought to give rise to Galois connections, and showed that it is possible to derive mass noun denotations for time points from those for time intervals, and vice versa. Our approach allows one to talk rigorously about changes of states and about events of continuous nature. In particular, we argued that sentences of continuous production or consumption inherently require mathematical integration. Since our theory is framed in terms of measure theory, they receive a natural treatment with Lebesgue integration. In fact, regardless of whether or not mass nouns (or events of continuous nature) are involved, it is expected that a great deal of existential quantification may be eliminated from the truth conditions of natural language sentences in favor of conditions on Lebesgue integrals; one would only need to invoke appropriate measures for involved nouns and verbs. Finally, we sketched how we might develop a theory of telicity on our approach. I hope to have demonstrated both the merit and necessity of a measure-theoretic approach to natural language semantics.

Obviously, we have only scratched the surface of this new direction of development, and much remains to be investigated and to be elaborated upon. For instance, we treated nouns and verbs as denoting relations between times and entities, but a more precise analysis should include eventualities. In such an analysis, mass nouns will be associated with states, and verbs of production or consumption with processes of some sort. The times that we have so far associated with nouns and verbs will correspond to (the equivalence classes of) the projections of eventualities onto the time axis via something like Krifka's (1989) temporal trace function. The different properties exhibited by denotations of nouns and verbs should then be ascribed to the different types of underlying eventualities. Also, while the Lebesgue-integral method yields a satisfactory treatment as far as meaning is concerned, it gives one a good deal of headache trying to work out how all this may be achieved compositionally.

Finally, a note on possible worlds is in order. So far, we have fixed the world parameter and focused exclusively on temporal changes, but much similar development is expected for worlds as well. If we trace our train of discussion, we should be dealing mainly with (equivalence classes of) sets of possible worlds of positive measure rather than with possible worlds *per se*, just as we have decided to deal mainly with sets of time points of positive measure rather than time points. Those sets of possible worlds could be viewed as representing partial information of a world, so they might be comparable to situations (Barwise and Perry, 1983; Kratzer, 1989). The obvious candidate for the measure on these "situations" will be one that assigns them their probabilities. By positing a "situational trace function" similar to the temporal trace function, predicates could be analyzed as relations between situations and entities, if the temporal parameter is fixed. Lebesgue integrals over situations will calculate expected values. All of this is mere speculation at this point, but I hope to delve deeper into these and other matters in future research.

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