The Computational Cost of Quantifier Raising

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Abstract

Quantifier Raising (QR) is a tremendously popular tool for repairing type mismatches and analyzing scope in LF-based approaches to semantic interpretation. But despite its prominence, its computational properties are barely known. To fill this gap, I study the time complexity of the following decision problem: given a tree whose leaves are assigned semantic types, can repeated applications of QR lead to a well-typed logical form (LF)? The main result is that this problem is NP-complete, meaning that there exists no general and efficient algorithm for building interpretable LFs using QR (assuming P $\neq$ NP). The problem remains NP-complete even if one constrains QR by limiting the type of traces, limiting the number of atomic semantic types, banning parasitic scope, and banning cyclic QR.

1 Introduction

Any compositional semantic analysis has to answer two questions: what are the meaningful pieces, and how do they fit together? The need for answers becomes both pressing and difficult as soon as a syntactic structure contains type mismatches, or a subexpression has to take scope over a portion of its context. Semanticists have met these challenges over the years with a battery of methods like Quantifier Raising (May, 1977; Heim and Kratzer, 1998), Quantifying In (Montague, 1973), Cooper Storage (Cooper, 1983), Continuation-Passing Style (Barker and Shan, 2014; Kiselyov and Shan, 2014), and Flexible Montague Grammar (Hendriks, 1993). Varied as they are, at a high level these tools are all just ways of chopping up trees and piecing them back together in a fashion that makes the computation of meaning possible. One way or another, every compositional theory of semantic interpretation ends up sawing and gluing.

This paper studies that process of semantic reshuffling by taking a close look at Quantifier Raising. QR is a particularly simple and popular way to solve the problem of type mismatches and displaced scope, but despite fifty years of widespread use, surprisingly little is known about it computationally. In order to improve our formal understanding, I therefore answer some basic but unresolved questions about its complexity:

- Given a logical form that is *not* well-typed, how difficult is it to determine whether repeated applications of QR can turn it into one that *is*?
- Does the difficulty depend on the number of atomic types one uses? Does it depend on the possibility of higher-order traces, or the possibility of parasitic scope, or the possibility of QR applying to an element multiple times?
- Do we need any purely formal constraints on quantifier raising to keep it well-behaved?

My answer to the first question above is: *very* difficult (NP-complete). My answer to the second is: *no*, none of those constraints affects the difficulty (as measured by asymptotic, worst-case time complexity). My answer to the third question is: *yes*, we need a formal constraint that prevents QR from targeting λ-abstractions that were themselves created by QR.

To date, the only formal study of QR is Barker (2020). This paper looks at the operation in the same spirit that Barker did and builds on the foundation he laid.

The rest of the paper is organized as follows. Section 2 provides a formal definition of logical forms, of Quantifier Raising, and of the decision problem LF REPAIR: given a possibly ill-typed logical form, can repeated applications of QR make it well typed? The section also proves a couple of useful facts about logical forms. Section 3 proves that LF REPAIR belongs to the complexity class NP, following Barker (2020) closely. Section 4 proves that

LF REPAIR is NP-hard by a reduction from the DIRECTED HAMILTONIAN CYCLE problem. Section 5 makes a few concluding remarks.

I assume that the reader is familiar with basic notions from computational complexity theory, in particular the theory of NP-completeness. For a general introduction, see Papadimitriou (1994) and Arora and Barak (2009). For overviews focused on linguistic issues, see Pratt-Hartmann (2010) and Barton et al. (1987).

2 Preliminaries and Notation

2.1 Types, Logical Forms, Contexts

Given a finite set $\mathbf{A}$ of atomic types, the set of *types* over it is

$$\mathbf{T} ::= \mathbf{A} \mid \mathbf{T} \to \mathbf{T}.$$

In the rest of the paper, lowercase greek letters $(\alpha, \beta, \gamma, \ldots)$ will range over types. As usual, '$\to$' associates to the right. I often drop the '$\to$', so that $\alpha\beta\gamma$, for example, is a shorthand for the type $\alpha \to (\beta \to \gamma)$. I use $\alpha \xrightarrow{m} \beta$ as a shorthand for

$$\underbrace{\alpha \to \cdots \to \alpha}_{m \text{ times}} \to \beta$$

A useful way to stratify the set of types is by assigning each one an *order*. If we think of types as describing functions, then the order quantifies how complex that function's arguments are. Formally, the order of an atomic type p is $\operatorname{ord}(p) = 1$, and the order of a complex type $\alpha \to \beta$ is

$$\operatorname{ord}(\alpha \to \beta) = \max(1 + \operatorname{ord}(\alpha), \operatorname{ord}(\beta)).$$

Types of order n describe functions whose arguments have types of order less than n. Some examples: if $e, t \in \mathbf{A}$, then $\operatorname{ord}(e) = 1$, $\operatorname{ord}(eet) = 2$, and $\operatorname{ord}\big((et)et\big) = 3$.

Let $\{\mathbf{V}_\alpha\}_{\alpha \in \mathbf{T}}$ be a disjoint family of countable sets indexed by $\mathbf{T}$, and let

$$\mathbf{V} = \bigcup_{\alpha \in \mathbf{T}} \mathbf{V}_\alpha.$$

The members of each set $\mathbf{V}_\alpha = \{x, y, \ldots\}$ are *variables of type* α. Nodding at linguistic practice, I sometimes call variables *traces*. When the type of a variable x is relevant, I will write it as x^α to indicate that $x \in \mathbf{V}_\alpha$.

The set $\mathbf{L}$ of *logical forms* is

$$\mathbf{L} ::= \mathbf{T} \mid \mathbf{V} \mid \mathbf{L} \bullet \mathbf{L} \mid \lambda\mathbf{V} \circ \mathbf{L}.$$

In words, an LF is either a type, a variable, the concatenation of two LFs, or a λ-abstraction over an LF. An *abstraction-free* LF is one built from only the first three cases, i.e., an LF that contains no λ-abstractions. It's usual in semantics to place typed lexical items rather than types themselves at the leaves of logical forms. But since this paper is only concerned with type coherence, the definition above is simpler.

The constructor '$\bullet$' associates to the left and '$\circ$' to the right. Both are commutative, meaning $\alpha \bullet \beta$ and $\beta \bullet \alpha$ are considered equivalent, as are $\alpha \circ \beta$ and $\beta \circ \alpha$. '$\circ$' binds more tightly than '$\bullet$'.

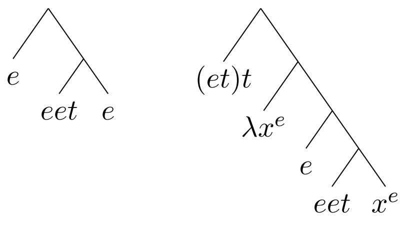
If $e, t \in \mathbf{A}$, then

$$e \bullet (eet \bullet e) \quad \text{and} \quad (et)t \bullet \lambda x^e \circ (e \bullet (eet \bullet x^e))$$

are each examples of logical forms. From now on I'll leave out the '$\circ$' after λ-abstractions, representing for instance the second LF above as

$$(et)t \bullet \lambda x^e(e \bullet (eet \bullet x^e)).$$

I sometimes present LFs as unordered binary trees in a way familiar to linguists, e.g.



Later on I will need to talk about LFs with a "hole" that can be plugged by another LF. I call these *contexts*. Formally, the set of contexts is

$$\mathbf{C} ::= [\,] \mid \mathbf{L} \bullet \mathbf{C} \mid \lambda\mathbf{V} \circ \mathbf{C}.$$

If Γ is a context and Δ is an LF (or context), then $\Gamma[\Delta]$ is the LF (or context) that results from substituting Δ for $[\,]$ in Γ. In Section 4, it will be useful to have notation for repeatedly nesting a context inside itself: if Γ is a context, let $\Gamma^0 = [\,]$ and $\Gamma^n = \Gamma[\Gamma^{n-1}]$. In the rest of the paper, capital Greek letters (Γ, Δ, etc.) will denote either LFs or contexts.

The *typing function* $\tau : \mathbf{L} \to \mathbf{T}$ is the following partial map:

$$\tau(\Gamma) = \begin{cases} \alpha & \text{if } \Gamma = \alpha, \\ \beta & \text{if } \Gamma = \Delta \bullet \Theta, \tau(\Delta) = \alpha\beta \\ & \quad \text{and } \tau(\Theta) = \alpha, \\ \alpha\beta & \text{if } \Gamma = \lambda x^\alpha\, \Delta[x^\alpha] \text{ and} \\ & \quad \tau(\Delta[\alpha]) = \beta, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

The second and third cases correspond respectively to the semantic rules of *function application* and *predicate abstraction* in Heim and Kratzer (1998). When $\tau(\Gamma) = \alpha$ holds, I will usually just say Γ has type α. I occasionally use the notation $\Gamma : \alpha$ to mean that either $\tau(\Gamma) = \alpha$ or $\Gamma = x$ for some $x \in \mathbf{V}_\alpha$. (Note that $\tau(x^\alpha)$ is formally undefined.)

An LF Γ is *well typed* iff $\tau(\Gamma)$ is defined. The two LFs mentioned above, $e \bullet (eet \bullet e)$ and $(et)t \bullet \lambda x^e\,(e{\bullet}(eet{\bullet}x^e))$, each have type t and are therefore well typed.

A trace x in an LF is *bound* if it occurs within a subtree of the form $\lambda x\, \Sigma[x]$. Otherwise it's *unbound*. Notice that an LF containing an unbound trace cannot be well typed. The definition of the typing function therefore enforces a form of the empirically motivated proposal that remnant movement must reconstruct (Huang, 1993; Bhatt and Dayal, 2007): movement steps that create an unbound trace are always "undone" before interpretation begins.

2.2 Quantifier Raising

The surface syntax of a sentence is often ill-typed and therefore not suitable for semantic interpretation. A classic example comes from transitive sentences with a quantifier in object position ("the critic panned every film"), which correspond to the ill-typed LF $e \bullet (eet \bullet (et)t)$. Function application can't combine the type of a transitive verb, eet, with the type of a generalized quantifier, $(et)t$.

In this case, we know what LF we would *like* to interpret (Heim and Kratzer, 1998, 178–79). The proper semantic argument for the quantifier in object position is an abstraction over its context, also known as its continuation, so the LF we want is

$$\underbrace{(et)t}_{\text{quantifier}} \bullet \underbrace{\lambda x^e\,(e \bullet (eet \bullet x^e))}_{\text{continuation}}.$$

Quantifier Raising is just an operation for moving from ill-typed LFs like $e \bullet (eet \bullet (et)t)$ to the well-typed one above. Given an LF $\Gamma[\Delta[\Theta]]$, QR "raises" Θ so that it becomes Δ's sister, puts a variable in Θ's old position, and adds a binder for that variable between Θ and Δ. The result is $\Gamma[\Theta \bullet \lambda x\, \Delta[x]]$. In the case above,

$$\Gamma = [\,],\;\; \Delta = e \bullet (eet \bullet [\,]),\;\; \Theta = (et)t.$$

My goal is to study QR in its most general form. Accordingly, I won't constrain it by—just to name a few possibilities—imposing scope islands, scope economy (Fox, 2000), or limits on the types of traces (Poole, 2024). Whatever empirical merits they might have, these restrictions would only weaken the mathematical results in this paper. More importantly, to understand the effect (if any) that specific constraints have on complexity, it's necessary to first understand the complexity of the general case.

That said, *one* formal constraint is necessary to keep QR from running totally amok: the operation should be disallowed from targeting abstractions, that is, LFs of the form $\lambda x\, \Gamma$. To illustrate what can go wrong without this constraint, look at Figure 1. The leftmost tree shows an innocent-looking analysis of the sentence *Mary gave John the book*. It might not seem like QR can do anything interesting to this tree, because no subtree has a complex enough type to take scope. But the ability to raise abstractions lets us create a scope-taker.

The middle tree is the result of applying QR to *gave*, moving it to a position right below *Mary* and leaving a trace of type eet. The rightmost tree is the result of quantifier raising the *abstraction* created by the previous step to the root of the tree and leaving a trace of type e. Not only is the resulting LF well typed, but (assuming natural denotations for the lexical items) it means *the book gave John Mary*.

Completely unrestricted QR therefore allows us to create scope-takers on the fly, even in sentences where nothing ought to be taking scope, and the semantic effect is disastrous. It does not behave in the way linguists expect QR to behave and it lacks good theoretical properties. The definition below therefore explicitly prevents QR from targeting abstractions.[1]

Definition 2.1. *Quantifier Raising* is the smallest binary relation $\rightarrow_{\text{QR}} \subseteq \mathbf{L} \times \mathbf{L}$ such that for all contexts Γ and Δ, all LFs Θ that are *not* abstractions, and all fresh variables x,

$$\Gamma[\Delta[\Theta]] \rightarrow_{\text{QR}} \Gamma[\Theta \bullet \lambda x\, \Delta[x]].$$

The reflexive, transitive closure of $\rightarrow_{\text{QR}}$ is $\rightarrow^*_{\text{QR}}$.

[1]A more elegant way to ban QR of abstractions is to change the structure of LFs so that no constituent corresponds to an abstraction in the first place. Variable binding would instead be handled by an index on moved items, so that QR would relate an LF $\Gamma[\Delta[\Theta]]$ to something like $\Gamma[\Theta_x \bullet \Delta[x]]$ (Heim, 1997; Heim and Kratzer, 1998, 187–88). But this approach can't easily accommodate *parasitic scope* (Barker, 2007). Since I want a formalization of quantifier raising that covers all of its uses in linguistics, I've therefore stuck with tradition and used λ-abstraction to bind variables.

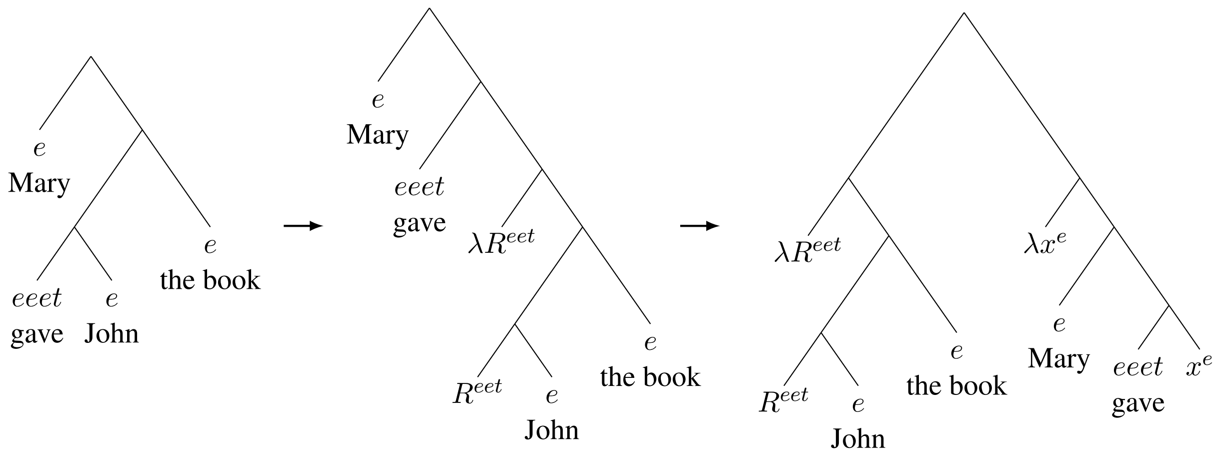


Figure 1: Raising abstractions is dangerous.

Because this operation targets not just quantifiers but any part of an LF at all, the name "Quantifier Raising" is misleading. "Scope Taking," as Barker (2020) says, is more accurate. But like him, I've stuck with the traditional, firmly entrenched name.

Now we can define the central decision problem of this paper:

> LF REPAIR
> Given a set of atomic types $\mathbf{A}$ and an LF Γ, is there an LF Δ of type α such that $\Gamma \rightarrow^*_{\text{QR}} \Delta$?

The name comes from the idea that ill-typed LFs are "broken," and that repairing them involves raising the scope-takers and finding appropriate trace types in such a way that the result is well-typed.

Here is an example. Let $\mathbf{A} = \{e, t\}$ and let Γ be the ill-typed LF

$$\big[(et)e \bullet \big[(((et)et)et)et \bullet et\big]\big] \bullet \big[eet \bullet (et)t\big].$$

Is there a Δ of type t such that $\Gamma \rightarrow^*_{\text{QR}} \Delta$? The answer, not immediately obvious, is yes:

$$\begin{aligned}\Gamma &\rightarrow_{\text{QR}} (et)t \bullet \lambda x^e\big[\big[(et)e \bullet \big[(((et)et)et)et \bullet et\big]\big] \\ &\qquad \bullet \big[eet \bullet x^e\big]\big] \\ &\rightarrow_{\text{QR}} (et)t \bullet \big[(((et)et)et)et \bullet \lambda Q^{(et)et} \lambda x^e \\ &\qquad \big[\big[(et)e \bullet \big[Q^{(et)et} \bullet et\big]\big] \bullet \big[eet \bullet x^e\big]\big]\big],\end{aligned}$$

and this last LF has type t.

2.3 Normalization

QR's purpose is to let a function take some portion of its semantic context as an argument. But it's also possible for the raised element to become an argument rather than a function of its context: for example, $e \bullet et \rightarrow_{\text{QR}} e \bullet \lambda x^e(x^e \bullet et)$. Such steps are licit but pointless—they will never help turn an ill-typed LF into a well-typed one. In general, if an LF has the form $\Delta[\Pi \bullet \lambda x\, \Sigma[x]]$, where $\Pi : \alpha$ and $\lambda x\, \Sigma[x]$ has type $\alpha\beta$, then I call the abstraction $\lambda x\, \Sigma[x]$ *vacuous*. A well-typed LF is *normalized* iff it contains no vacuous abstractions.

The next lemma justifies the label "vacuous."

Lemma 2.1. *Let Γ be a variable- and abstraction-free LF. If $\Gamma \rightarrow^*_{\text{QR}} \Delta$, where Δ has type α, then there exists a normalized LF Δ' of type α such that $\Gamma \rightarrow^*_{\text{QR}} \Delta'$.*

Proof. If Δ isn't already normalized, then it contains a vacuous abstraction and therefore has the form $\Theta[\Pi \bullet \lambda x^\beta\, \Sigma[x^\beta]]$, where $\Pi : \beta$ and $\lambda x^\beta\, \Sigma[x^\beta]$ has type $\beta\gamma$. By definition, QR cannot target an abstraction. And since Δ is well-typed, no step in the derivation $\Gamma \rightarrow^*_{\text{QR}} \Delta$ raises x^β outside the scope of its binder λx^β. The derivation $\Gamma \rightarrow^*_{\text{QR}} \Delta$ therefore has the following form:

$$\begin{aligned}\Gamma &\rightarrow^*_{\text{QR}} \Theta_1[\Sigma_1[\Pi_1]] \\ &\rightarrow_{\text{QR}} \Theta_1[\Pi_1 \bullet \lambda x^\beta\, \Sigma_1[x^\beta]] \\ &\rightarrow_{\text{QR}} \Theta_2[\Pi_2 \bullet \lambda x^\beta\, \Sigma_2[x^\beta]] \\ &\rightarrow_{\text{QR}} \cdots \\ &\rightarrow_{\text{QR}} \Theta_n[\Pi_n \bullet \lambda x^\beta\, \Sigma_n[x^\beta]],\end{aligned}$$

where the Θ_i and Σ_i are contexts; the Π_i are LFs; and $\Theta_n = \Theta$, $\Sigma_n = \Sigma$, and $\Pi_n = \Pi$. Then

$$\begin{aligned}\Gamma &\rightarrow^*_{\text{QR}} \Theta_1[\Sigma_1[\Pi_1]] \\ &\rightarrow_{\text{QR}} \Theta_2[\Sigma_2[\Pi_2]] \\ &\rightarrow_{\text{QR}} \cdots \\ &\rightarrow_{\text{QR}} \Theta_n[\Sigma_n[\Pi_n]]\end{aligned}$$

is a valid derivation of $\Theta[\Sigma[\Pi]]$ from Γ. Like Δ, the LF $\Theta[\Sigma[\Pi]]$ also has type α, but it contains one fewer vacuous abstraction.

By repeating this process, we can eliminate vacuous abstrations one by one until we obtain a normalized LF Δ' of type α such that $\Gamma \to^*_{\text{QR}} \Delta'$. □

The following lemma, needed in Section 4, makes it easier to check whether a variable- and abstraction-free LF Γ can be repaired. It turns out that the order of the traces created by each QR step never has to exceed a constant fixed by Γ.

Lemma 2.2. *Suppose Γ is a variable- and abstraction-free LF, Δ is a normalized LF, and $\Gamma \to^*_{\text{QR}} \Delta$. Let m be the maximum order among Γ's leaves. If x^α is a trace occurring in Δ, then* $\operatorname{ord}(\alpha) < m - 1$.

Proof. Let n be the maximum order among all of Δ's traces, and suppose $n \geq m - 1$. Consider a trace x^α of order n and write

$$\Delta = \Theta[\Pi \bullet \lambda x^\alpha\, \Sigma[x]].^{2}$$

Because Δ is normalized, Π's type has the form $(\alpha \to \tau(\Sigma[\alpha])) \to \beta$ for some β, and so

$$\operatorname{ord}(\tau(\Pi)) > \operatorname{ord}(\alpha) + 1 = n + 1. \qquad (1)$$

Every node within the subtree Π is either (i) a leaf of order at most $m \leq n + 1$, (ii) function application of two elements of order at most $n + 1$, or (iii) an abstraction over a variable of order at most n. It follows (by an induction from the leaves to the root) that every node of Π has order at most $n + 1$, so $\operatorname{ord}(\tau(\Pi)) \leq n + 1$. This contradicts (1). Therefore $n < m - 1$. □

3 LF REPAIR is in NP

This goal of this section is to prove that LF REPAIR is in NP. This result would be a simple corollary of the proof in Barker (2020) that LF REPAIR is decidable, but it turns out the proof is not valid.[3] The rest of this section provides a corrected proof, and uses it to show that LF REPAIR is in NP. Because many of the details from Barker's paper don't need to change, the proofs in this section move fast; see his paper for further information.

The plan, following Barker's lead, is to define a type-logical grammar called QRT (Quantifier Raising with Types) that captures the logic of quantifier raising in the following sense: if Γ is an LF, then $\Gamma \vdash A$ is a theorem of QRT if and only if $\Gamma \to^*_{\text{QR}} \Delta$ for some Δ of type A.[4] I show below that provability in QRT is in NP. Since LF REPAIR clearly has the same time complexity as provability in QRT, it must also be in NP.

The axiom and inference rules of QRT are shown in Figure 2. Γ, Δ and Θ represent LFs or contexts, and A and B represent types. $\lambda^\uparrow$ and $\lambda^\downarrow$ are subject to the restriction that the LF Θ not be a λ-abstraction (or equivalently, that Θ be either a type or of the form $\Sigma \bullet \Omega$). This restriction, which is the main difference between the version of QRT in this paper and the one in Barker (2020), will guarantee that the critical Theorem 3.2 holds. Unrestricted raising—which is what caused problems in Barker's proof—wreaks havoc when combined with the commutativity of '•'. For instance, it lets us freely swap an inner and outer context above a '•'-node, which results in massive overgeneration:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma[\Delta[A \bullet B]] \vdash C}{\Gamma[A \bullet \lambda x\, \Delta[x \bullet B]] \vdash C}\lambda^\downarrow}{(\lambda x\, \Delta[x \bullet B]) \bullet \lambda y\, \Gamma[A \bullet y] \vdash C}\lambda^\downarrow}{\Delta[(\lambda y\, \Gamma[A \bullet y]) \bullet B]) \vdash C}\lambda^\uparrow}{\Delta[\Gamma[A \bullet B]] \vdash C}\lambda^\uparrow$$

This issue is not really different from the one in Figure 1, and my solution is similar: forbidding $\lambda^\uparrow$ and $\lambda^\downarrow$ from targeting abstractions.

Figure 3 shows a small QRT proof.

There is a natural decision problem associated with QRT. As mentioned above, it has exactly the same time complexity as LF REPAIR.

> QRT PROVABILITY
> Given a sequent, is it a theorem of QRT?

The next theorem shows that cut-elimination is possible in proofs of QRT sequents that are *abstraction-free*, that is, sequents $\Gamma \vdash A$ where Γ contains no λ-abstractions.

Theorem 3.1. *Every abstraction-free theorem of QRT has a* CUT*-free proof.*

Proof. By the normal method of permuting cuts upward. When the cut formula is not principal in

[2]Technically, there could be more abstractions intervening, so that I should write $\Theta[\Pi \bullet \lambda y_1^{\beta_1} \ldots \lambda y_k^{\beta_k}\, \lambda x^\alpha\, \Sigma[x^\alpha]]$. But this doesn't affect the rest of the proof.

[3]Barker points out the problem himself in a footnote added after the early-access version of the paper was published.

[4]For a proof which carries over with minimal changes to the version of QRT in this paper, see the appendix to Barker (2020).

$$\dfrac{}{A \vdash A}\,\textsc{Ax} \qquad \dfrac{\Gamma \vdash A \qquad \Delta[B] \vdash C}{\Delta[AB \bullet \Gamma] \vdash C}\,{\to}\mathrm{L} \qquad \dfrac{\Gamma[A] \vdash B}{\lambda x^{A}\, \Gamma[x^{A}] \vdash AB}\,{\to}\mathrm{R}$$

$$\dfrac{\Gamma[\Theta \bullet \lambda x\, \Delta[x]] \vdash A}{\Gamma[\Delta[\Theta]] \vdash A}\,\lambda^{\uparrow} \qquad \dfrac{\Gamma[\Delta[\Theta]] \vdash A}{\Gamma[\Theta \bullet \lambda x\, \Delta[x]] \vdash A}\,\lambda^{\downarrow} \qquad \dfrac{\Gamma \vdash A \qquad \Delta[A] \vdash B}{\Delta[\Gamma] \vdash B}\,\textsc{Cut}$$

Figure 2: Axiom and Rules of QRT.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{e \vdash e}\,\textsc{Ax} \qquad \dfrac{\dfrac{}{et \vdash et}\,\textsc{Ax} \qquad \dfrac{}{t \vdash t}\,\textsc{Ax}}{(et)t \bullet et \vdash t}\,{\to}\mathrm{L}}{(et)t \bullet (eet \bullet e) \vdash t}\,{\to}\mathrm{L}}{\lambda x^{e}\, ((et)t \bullet (eet \bullet x^{e})) \vdash et}\,{\to}\mathrm{R} \qquad \dfrac{}{t \vdash t}\,\textsc{Ax}}{\dfrac{(et)t \bullet \lambda x^{e}\, ((et)t \bullet (eet \bullet x^{e})) \vdash t}{(et)t \bullet (eet \bullet (et)t) \vdash t}\,\lambda^{\uparrow}}\,{\to}\mathrm{L}$$

Figure 3: An example QRT derivation.

both premises of CUT, the proof transformations are straightforward.

The subtle case is when the cut formula *is* principal in both premises:

$$\dfrac{\dfrac{\Gamma[A] \vdash B}{\lambda x\, \Gamma[x] \vdash AB}\,{\to}\mathrm{R} \qquad \dfrac{\Delta \vdash A \qquad \Theta[B] \vdash C}{\Theta[AB \bullet \Delta] \vdash C}\,{\to}\mathrm{L}}{\Theta[(\lambda x\, \Gamma[x]) \bullet \Delta] \vdash C}\,\textsc{Cut}$$

Notice that Δ cannot be an abstraction. To see why, first define a sequent to be *stuck* if its left-hand side has the form $\Sigma[\Phi \bullet \Psi]$ where Φ and Ψ are both abstractions. A straightforward induction shows that if a sequent in a QRT proof is stuck, then every subsequent sequent—and in particular the conclusion—is stuck too. Now if Δ *were* an abstraction, then $\Theta[(\lambda x\, \Gamma[x]) \bullet \Delta] \vdash C$ would be stuck and the proof's conclusion would contain an abstraction, contrary to assumption. Δ is therefore not an abstraction, and we can replace the cut with two smaller ones:

$$\dfrac{\dfrac{\dfrac{\Delta \vdash A \qquad \Gamma[A] \vdash B}{\Gamma[\Delta] \vdash B}\,\textsc{Cut} \qquad \Theta[B] \vdash C}{\Theta[\Gamma[\Delta]] \vdash C}\,\textsc{Cut}}{\Theta[(\lambda x\, \Gamma[x]) \bullet \Delta] \vdash C}\,\lambda^{\downarrow}$$

□

Theorem 3.2. *Every abstraction-free theorem of QRT has a* CUT- *and* $\lambda^{\downarrow}$*-free proof.*

Proof. Take a CUT-free proof of an abstraction-free theorem, and consider an occurrence of $\lambda^{\downarrow}$ of maximal depth. Because the proof's conclusion is abstraction-free, there must be a lower occurrence of $\lambda^{\uparrow}$ eliminating the abstraction introduced by $\lambda^{\downarrow}$. If the occurrence of $\lambda^{\uparrow}$ is immediately below, then the application of $\lambda^{\downarrow}$ is eliminable:

$$\dfrac{\dfrac{\Gamma[\Delta[\Theta]] \vdash A}{\Gamma[\Theta \bullet \lambda x\, \Delta[x]] \vdash A}\,\lambda^{\downarrow}}{\Gamma[\Delta[\Theta]] \vdash A}\,\lambda^{\uparrow} \quad \rightsquigarrow \quad \Gamma[\Delta[\Theta]] \vdash A$$

Otherwise we can permute the occurrence of $\lambda^{\downarrow}$ downward across any occurrence of $\to$L, $\to$R, and $\lambda^{\uparrow}$ until it meets up with a suitable lower occurrence of $\lambda^{\uparrow}$ and can be removed. (The requirement that $\lambda^{\uparrow}$ and $\lambda^{\downarrow}$ not target abstractions guarantees that it is possible to permute $\lambda^{\downarrow}$ across $\lambda^{\uparrow}$ whenever needed.)

By repeating this procedure, we can eliminate each occurrence of $\lambda^{\downarrow}$ one by one. Since the procedure doesn't introduce any new applications of CUT, the end result is a CUT- and $\lambda^{\downarrow}$-free proof. □

Theorem 3.3. QRT PROVABILITY *is in NP.*

Proof. A sequent $\lambda x^{A}\, \Gamma[x^{A}] \vdash AB$ is provable if and only if $\Gamma[A] \vdash B$ is, and $\Gamma[\Sigma \bullet \lambda x\, \Delta[x]] \vdash A$ is provable if and only if $\Gamma[\Delta[\Sigma]] \vdash A$ is. Given an arbitrary sequent, we can use these facts to efficiently find an *abstraction-free* sequent that is provable iff the original sequent is. QRT-provability for arbitrary sequents is therefore reducible in polynomial time to QRT-provability of *abstraction-free* sequents, which we now show is in NP.

Theorem 3.2 shows that every abstraction-free theorem of QRT has a proof using only the rules Ax, $\rightarrow$L, $\rightarrow$R, and $\lambda^{\uparrow}$. Read bottom to top, each application of $\rightarrow$L and $\rightarrow$R eliminates an occurrence of '$\rightarrow$', and no rule introduces new ones. The number of uses of each of these rules in a proof is therefore bounded by the number of '$\rightarrow$'s in the conclusion. Since each application of $\lambda^{\uparrow}$ introduces a λ-abstraction that must be removed by a corresponding application of $\rightarrow$R, it can only be used as many times as $\rightarrow$R is. Every abstraction-free theorem of QRT therefore has a proof whose size is polynomial in the theorem's length and which can serve as an efficient witness of theoremhood. $\square$

Since LF REPAIR has the same time complexity as provability in QRT, the result we set out to show follows:

Theorem 3.4. LF REPAIR *is in NP.*

4 LF REPAIR is NP-Hard

4.1 Directed Graphs and Hamiltonian Cycles

The proof in this section uses some terminology from graph theory. As a reminder, a directed graph (or digraph) is a pair $\langle V, E\rangle$, where V is a finite set of *vertices* and $E \subseteq V \times V$ a set of *edges*. The first component of an edge is its *tail*, the second its *head*. Given a vertex v, its *indegree* $\deg^{-}(v)$ and *outdegree* $\deg^{+}(v)$ are the number of edges of which v is a head and tail, respectively.

A *Hamiltonian cycle* in a digraph is a sequence of edges $e_1, e_2, \ldots, e_n$ such that for $0 < i < n$, the head of e_i is the tail of e_{i+1}, the head of e_n is the tail of e_1, and every vertex occurs exactly once as a head and a tail of an edge in the sequence. The following decision problem is NP-complete (see, e.g., Garey and Johnson 1979):

> DIRECTED HAMILTONIAN CYCLE
> Given a digraph, does it have a Hamiltonian cycle?

4.2 Encoding Hamiltonian Cycles in LFs

This section describes a polynomial-time reduction from DIRECTED HAMILTONIAN CYCLE to LF REPAIR. Let $G = \langle V, E\rangle$ be a digraph with $n > 0$ vertices and $m \geq n$ edges. It's convenient to assume that the vertices of G are some initial segment of the natural numbers, i.e. $V = \{1, \ldots, n\}$. I will construct an instance of LF REPAIR whose answer is "yes" iff G has a Hamiltonian cycle. The method has some similarities to the one in Krantz and Mobile (2001), who encode DIRECTED HAMILTONIAN CYCLE in the multiplicative fragment of linear logic.

The set of atomic types for the instance consists of a type v for each $v \in V$, plus the additional types a, b, c, d, e, and t. That is,

$$\mathbf{A} = \{1, \ldots, n\} \cup \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}, \mathsf{t}\}.$$

When describing the reduction below, I'll use the following abbreviations for types:

$$\begin{aligned}
\mathrm{E}_u^v &:= (\mathsf{a} \rightarrow u) \rightarrow \mathsf{b} \rightarrow v,\\
\mathrm{V}_v &:= (\mathsf{a} \rightarrow v) \rightarrow \mathsf{c} \rightarrow v,\\
\mathrm{G}_v &:= (\mathsf{a} \rightarrow \mathsf{t}) \rightarrow \mathsf{d} \rightarrow v,\\
\mathrm{G}^v &:= (\mathsf{a} \rightarrow v) \rightarrow \mathsf{e} \rightarrow \mathsf{t}.
\end{aligned}$$

The rest of this subsection shows how to build an LF Γ such that $\Gamma \rightarrow^{*}_{\mathrm{QR}} \Delta$ for some Δ of type t if and only if G has a Hamiltonian cycle. Without loss of generality, I assume the cycle begins at the vertex 1. To make the construction more concrete, I will illustrate it as I go using the digraph in Figure 4. Notice that the graph has a Hamiltonian cycle: $(1,3), (3,2), (2,4), (4,1)$.

I'll build Γ up in four layers, defining an LF Γ_1 and three contexts $\Gamma_2, \Gamma_3, \Gamma_4$, such that $\Gamma = \Gamma_4[\Gamma_3[\Gamma_2[\Gamma_1]]]$. The plan is as follows. Γ_1 will contain a leaf E_u^v for each edge (u, v) of G, as well as a leaf V_v for every vertex. Each one of these leaves will need to undergo QR to have a chance of making Γ into a well-typed LF. The context Γ_2 will have n positions to which n of the leaves E_u^v from Γ_1 can undergo QR, as well as n positions for the V_v. By design, the types will properly compose if and only if the n corresponding edges in G form a Hamiltonian cycle.

The $m - n$ leaves E_u^v corresponding to edges not occurring in the cycle will also need a place that they can QR to. The context Γ_4 will provide $m - n$ positions for them. Since a Hamiltonian cycle enters and exits every vertex exactly once, for each vertex v there are $\deg^{-}(v) - 1$ edges with head v and $\deg^{+}(v) - 1$ edges with tail v that do not occur in the cycle. Γ_3 will accordingly contain $\deg^{+}(v) - 1$ leaves G_v and $\deg^{-}(v) - 1$ leaves G^v. For each E_u^v that QRs into Γ_4, G_u and G^v will QR right below and above it in Γ_4. This last step is needed to make all the types combine.

Now for the technical details. Let $(u_1, v_1), \ldots, (u_m, v_m)$ be G's edges, and let

$$\Gamma_1 = (\mathsf{a} \xrightarrow{m+n} 1) \bullet \mathrm{E}_{u_1}^{v_1} \bullet \ldots \bullet \mathrm{E}_{u_m}^{v_m} \bullet \mathrm{V}_1 \bullet \ldots \bullet \mathrm{V}_n.$$

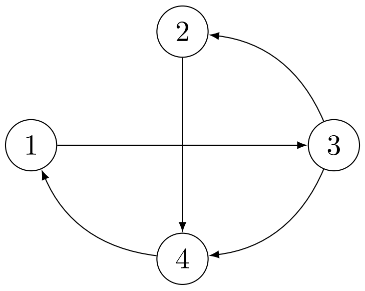
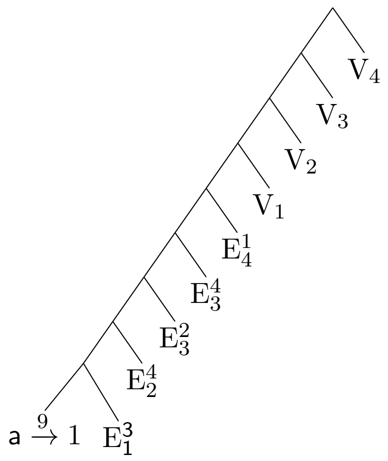


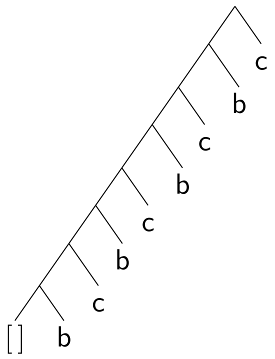
Figure 4: A directed graph.

For the digraph in Figure 4, Γ_1 would be



The intention is that every E_u^v and V_v will take scope and leave a trace of type a. The leaf of type $\mathsf{a} \xrightarrow{m+n} 1$ will consume these traces one by one, producing a subtree of type 1.

Let $\Delta = [\,] \bullet \mathsf{b} \bullet \mathsf{c}$ and let $\Gamma_2 = \Delta^n$. For the digraph in Figure 4, $\Gamma_2 = \Delta^4$, which looks like



If E_u^v and V_v each QR directly below the occurrences of b and c respectively in $\Delta[u]$, then the resulting subtree will have type v. Δ is therefore a gadget for simulating the traversal of an edge from u to v. The left side of Figure 5 illustrates the simulation with E_1^3 and V_3. Just for intuition, one can imagine that QR of E_u^v takes us from the state "exiting u" to the state "entering v", and QR of V_v takes us from the state "entering v" to the state "exiting v".

By nesting n copies of Δ, Γ_2 can simulate a path of length n in G. Notice that simulating the traversal of an edge with head v requires raising an occurrence of V_v. Since Γ will contain exactly one leaf V_v for each vertex v, the paths Γ_2 can simulate are those that visit each vertex exactly once, that is, Hamiltonian paths.

Γ_3 is most complicated. First define

$$a \dot{-} b = \max(a - b, 0).$$

Now for each $v \in V$ let

$$\Phi_v = [\,]\bullet \underbrace{\mathrm{G}_v \bullet \ldots \bullet \mathrm{G}_v}_{\deg^+(v) \dot{-} 1 \text{ times}}, \quad \Psi_v = [\,]\bullet \underbrace{\mathrm{G}^v \bullet \ldots \bullet \mathrm{G}^v}_{\deg^-(v) \dot{-} 1 \text{ times}}$$

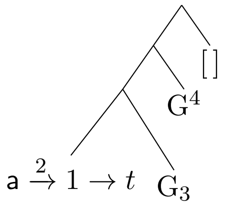
and let

$$\Phi = \Phi_1[\Phi_2[\ldots \Phi_n]] \quad \text{and} \quad \Psi = \Psi_1[\Psi_2[\ldots \Psi_n]].$$

Also let

$$k = \sum_{v=1}^{n} (\deg^+(v) \dot{-} 1) + (\deg^-(v) \dot{-} 1)$$

Then $\Gamma_3 = \Psi[\Phi[\mathsf{a} \xrightarrow{k} 1 \to t]] \bullet [\,]$. In the case of the digraph in Figure 4 (where only Φ_3 and Ψ_4 are nontrivial), Γ_3 would be



To make Γ well typed, each G_u and G^v will need to undergo QR and leave behind a trace of type a. The leaf of type $\mathsf{a} \xrightarrow{k} 1 \to t$ will consume these traces, resulting in a subtree of type $1 \to \mathsf{t}$.

Finally, let $\Sigma = [\,] \bullet \mathsf{d} \bullet \mathsf{b} \bullet \mathsf{e}$, and let $\Gamma_4 = \Sigma^{m-n}$. In the case of the digraph in Figure 4, Γ_4 would be

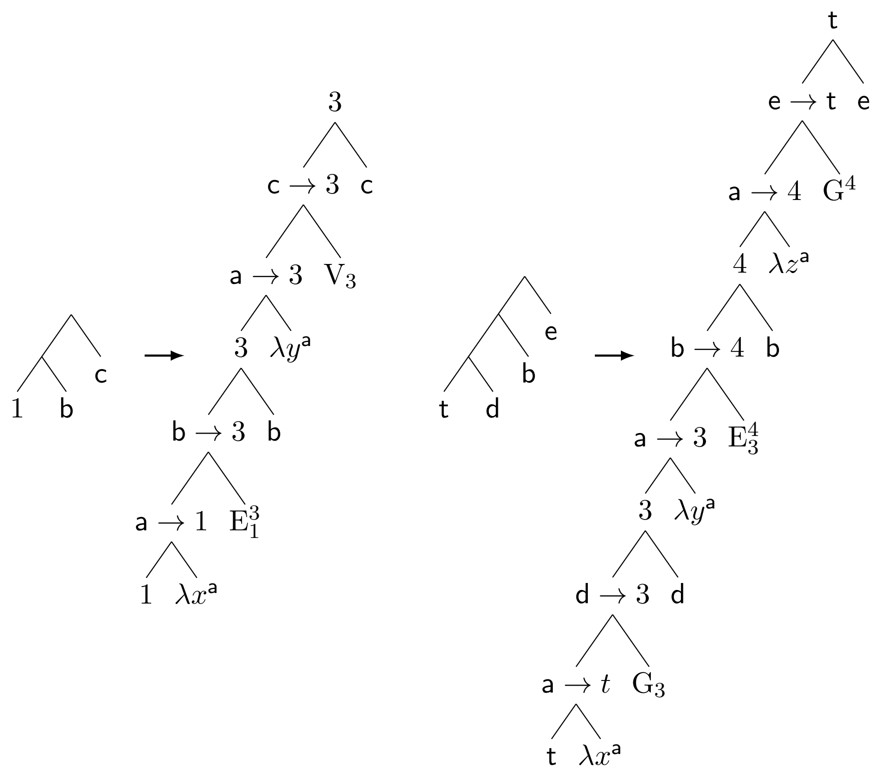
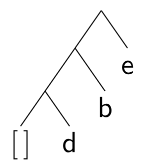


Figure 5: Simulating the traversal of an edge (left) and disposing of an unused edge (right). Interior nodes are annotated with the type of the subtree they dominate. λ-abstractions are now shown on the right, just for cosmetics (remember that LFs are *unordered* binary trees).



The context Σ is a gadget that works more or less like Δ. If G_u, E^v_u, and G^v each QR directly below the occurrences of d, b, and e respectively in $\Sigma[\mathsf{t}]$, the resulting subtree will have type t. The right side of Figure 5 illustrates the procedure with G_3, E^4_3, and G^4. Together $\Sigma[\mathsf{t}]$, G_u, and G^v work as a kind of disposal mechanism for the leaf E^v_u, providing it a place to which it can QR and returning a subtree of type t.

Γ_4 consists of $m-n$ nested copies of Σ. When G has a Hamiltonian cycle, Γ_1 will contain $m-n$ leaves E^v_u coming from edges not used in the cycle. Γ_4's purpose is to act as a waste bin of sorts for these unused edges. Notice that "disposing" of a leaf E^v_u requires raising both a leaf G_u and a leaf G^v from Γ_3. As mentioned earlier, for each vertex v there are $\deg^-(v)-1$ edges with head v and $\deg^+(v)-1$ edges with tail v that do not occur in the cycle. Γ_3 therefore provides exactly the right number of leaves G_u and G^v to dispose of all the unused edges.

Figure 6 shows the final LF $\Gamma = \Gamma_4[\Gamma_3[\Gamma_2[\Gamma_1]]]$. Notice that Γ will always be linear in the size of the input, and can be constructed in polynomial time.

4.3 Correctness

It remains to show that the reduction is correct, i.e., that G has a Hamiltonian cycle iff the answer to the LF REPAIR-instance constructed in the previous subsection is "yes".

First the easy direction.

Theorem 4.1. *If G has a Hamiltonian cycle, then there exists an LF Δ of type* t *such that $\Gamma \rightarrow^*_{\mathrm{QR}} \Delta$.*

Proof. Let

$$(u_1, u_2), (u_2, u_3), \ldots, (u_{n-1}, u_n), (u_n, u_1)$$

be a Hamiltonian cycle in G. Without loss of generality, assume that $u_1 = 1$. We can derive an appropriate Δ of type t from Γ as follows. Note that every QR step in the derivation below will leave a trace of type a.

First QR $\mathrm{E}^{u_2}_{u_1}$ and V_{u_2} directly below the lowest occurrence of b and c respectively in Γ_2 (compare the left side of Figure 5). Then QR $\mathrm{E}_{u_2}^{u_3}$ and V_{u_3} directly below the *second*-lowest b and c respectively in Γ_2. Continue this process for the rest of the cycle, finishing with $\mathrm{E}_{u_n}^{u_1}$ and V_{u_1}.

Exactly $m - n$ leaves of the form E_u^v in Γ_1 are still waiting to QR, each one of them corresponding to an edge not used in the Hamiltonian cycle. For each vertex v, exactly $\deg^-(v) - 1$ of these leaves correspond to an edge with head v and $\deg^+(v) - 1$ to an edge with tail v.

For each b in Γ_4, QR one of the $m-n$ remaining leaves E_u^v's directly below it. Then QR an instance of G_u and G^v from Γ_3 directly below the adjacent occurrences of d and e, respectively (compare the right side of Figure 5). Since the number of leaves G_u and G^v is $\deg^+(v) - 1$ and $\deg^-(u) - 1$ respectively, by the end of this process every G_u and G^v from Γ_3 will have undergone QR.

The LF resulting from these $3m-n$ applications of QR is the Δ we wanted. An easy check shows that it has type t. $\square$

Now for the harder direction, which comes down to showing that the procedure in the previous proof is the only way to make Γ well typed.

Theorem 4.2. *Suppose that* $\Gamma \to_{\mathrm{QR}}^{*} \Delta$ *for some* Δ *of type* t. *Then G has a Hamiltonian cycle.*

Proof. By Lemma 2.1, we can assume that Δ is normalized. We now make a series of claims about Δ. Below, I refer to a leaf that occurs in Γ as a "Γ-leaf."

(i) Every trace in Δ is of order one and therefore of atomic type. The claim follows from Lemma 2.2 and the fact that all of Γ's leaves have order at most three. A consequence is that nothing in the derivation $\Gamma \to_{\mathrm{QR}}^{*} \Delta$, undergoes QR more than once, since this would leave a trace of atomic type next to an abstraction, contradicting the fact that Δ is normalized.

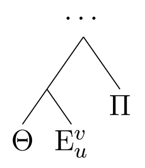
(ii) Every internal node of Δ has a type of order at most two. To prove the claim, suppose it were false and consider a maximally deep internal node of order greater than two. By (i), all abstractions are over variables of order one, so the node is not an abstraction (otherwise there would be a deeper internal node of the same order). It therefore has the form $\Theta \bullet \Lambda$, where Θ has type $\alpha\beta$, Λ has type α, and $\mathrm{ord}(\beta) > 2$. Since $\mathrm{ord}(\tau(\Theta)) = \mathrm{ord}(\alpha\beta) > 2$, Θ cannot be an internal node, so it must be a leaf. But each leaf of Δ is either a trace of order one or a Γ-leaf, and by inspection no Γ-leaf has the form $\alpha\beta$ where $\mathrm{ord}(\beta) > 2$. So no such internal node can exist.

(iii) Δ contains no "double abstractions," i.e., it is impossible to write $\Delta = \Theta[\lambda x\,(\lambda y\,\Lambda)]$. Suppose otherwise, and consider a "double abstraction" of *minimal* depth, so that we can write Δ as $\Theta[\Pi \bullet \lambda x^{\alpha}\,(\lambda y^{\beta}\,\Lambda)]$. Because Δ is normalized, Π must have a type of the form $(\alpha\beta\gamma)\delta$. This means that Π's type has order at least three, and so by (ii) Π must be a leaf. But no leaf of Δ has the form $(\alpha\beta\gamma)\delta$, a contradiction.

(iv) If an abstraction $\lambda x^{\alpha}\,\Pi$ occurs in Δ, then $\Delta = \Theta[l \bullet \lambda x^{\alpha}\,\Pi]$ for some context Θ and leaf l of type E_u^v, V_v, G_v, or G^v. This follows from (i), (ii), (iii), and the fact that Δ is normalized. A consequence is that in a derivation $\Gamma \to_{\mathrm{QR}}^{*} \Delta$, the only elements that undergo QR are leaves of type E_u^v, V_v, G_v, or G^v.

(v) Every trace in Δ has type a. If x^{α} is a trace in Δ, then it must be bound by some abstraction, so by (iv) we can write Δ as $\Theta[l \bullet \lambda x^{\alpha}\,\Pi[x^{\alpha}]]$, where l is a Γ-leaf. Since Δ is normalized, $l = (\alpha\beta)\gamma$ for some types β, γ. Inspecting the leaves of Γ shows that α can only be a.

(vi) Call the sister of the mother of a node in Δ its *aunt*. Then the aunt of each leaf E_u^v, V_v, G_v, and G^v in Δ is a Γ-leaf of type b, c, d or e, respectively. Consider, for example, a leaf E_u^v occurring in Δ with sister Θ and aunt Π, as shown below:



Θ would need order at least 4 to take

$$\mathrm{E}_u^v := (\mathsf{a} \to u) \to \mathsf{b} \to v$$

as an argument, but no node in Δ has order higher than 3. So Θ must have type $\mathsf{a} \to u$ and $\Theta \bullet \mathrm{E}_u^v$ must have type $\mathsf{b} \to v$. For Π to take $\mathsf{b} \to v$ as an argument, it would need to have order at least 3. The only such elements of Δ are the Γ-leaves, but no Γ-leaf has a type of the form $(\mathsf{b} \to v) \to \gamma$. So Π must have type b. Since all the traces in Δ have type a, the only nodes of type b in Δ are the m Γ-leaves coming from Γ_2 and Γ_4, and therefore Π is a Γ-leaf of type b, as we wanted to show. Similar arguments apply to the leaves of type V_v, G_v, and G^v. Notice that none of the leaves E_u^v, V_v, G_v, and G^v has an aunt of type b, c, d, or e in their base positions in Γ, so they must have all undergone QR in the derivation $\Gamma \to^*_{\text{QR}} \Delta$.

The upshot of all these observations is that in the derivation $\Gamma \to^*_{\text{QR}} \Delta$, every leaf of type E^v_u, V_v, G_v, and G^v undergoes QR *exactly once* (by (i) and (vi)); lands in a position directly below a unique leaf labeled b, c, d, or e respectively (by (vi)); leaves a trace of type a (by (v)); and *nothing else* undergoes QR (by (iv)).

We can now extract a Hamiltonian cycle from Δ. Exactly $2n$ leaves E^v_u and V_v in Δ underwent QR into Γ_2. Order them by decreasing depth:

$$\mathrm{E}^{v_1}_{u_1}, \mathrm{V}_{w_1}, \mathrm{E}^{v_2}_{u_2}, \mathrm{V}_{w_2}, \ldots, \mathrm{E}^{v_n}_{u_n}, \mathrm{V}_{w_n}.$$

In order for Δ to be well-typed, it must be the case that $u_1 = v_n = w_n = 1$ and

$$v_i = w_i = u_{i+1}, \quad 1 \leq i < n.$$

So we can rewrite the sequence above as

$$\mathrm{E}^{u_2}_{u_1}, \mathrm{V}_{u_2}, \mathrm{E}^{u_3}_{u_2}, \mathrm{V}_{u_3}, \ldots, \mathrm{E}^{u_1}_{u_n}, \mathrm{V}_{u_1}.$$

Since $V_{u_1}, \ldots, \mathrm{V}_{u_n}$ are pairwise distinct, so are $u_1, \ldots, u_n$, and

$$(u_1, u_2), (u_2, u_3), \ldots, (u_n, u_1),$$

is therefore a Hamiltonian cycle in G. $\square$

Putting Theorems 3.4, 4.1, and 4.2 together proves the main claim of this paper:

Theorem 4.3. LF REPAIR *is NP-complete.*

Section 3 mentioned that LF REPAIR has the same time complexity as provability in QRT. So a corollary of the above is:

Corollary 4.1. QRT PROVABILITY *is NP-complete.*

5 Conclusion

The main result of this paper is that finding well-typed logical forms using quantifier raising is NP-complete, and therefore belongs to a well-known class of intractable problems. Assuming P $\neq$ NP, this has an immediate practical consequence: there is no efficient program that takes in an input tree and computes the different scopal readings licensed by QR.

In this paper I've chosen, in part for mathematical convenience, to work with a particularly powerful version of QR. I've allowed it to raise almost any subtree to any position and to leave a trace of any type, so long as the final LF is well typed. A reader might wonder: does the intractability result depend on this choice? Not really. Even if one worked with a vastly weaker version of QR, the problem of repairing LFs would remain intractable. Just to give an example, notice that the reduction in Section 4

(i) did not need higher-order traces (in fact, every trace had type a),
(ii) did not need parasitic scope (Barker, 2007),
(iii) did not need any elements to undergo QR multiple times.

So even if one changed the definition of QR in Section 2 to always leave traces of a single atomic type, or to ban parasitic scope, or to disallow elements from undergoing QR multiple times, and even if one imposed all of these constraints simultaneously, the reduction in the previous section would still be valid and the problem of finding a well-typed LF would remain NP-complete. There may be good empirical arguments for or against (i)–(iii), but as far as computational tractability is concerned, none of them are consequential.

The reduction does help itself to an unbounded inventory of atomic types, but only a finite number are actually needed. Given the atomic types $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}, \mathsf{t}$ and an additional type v, it's possible to encode the types $1, \ldots, n$ in "unary": replace 1 with v, 2 with $\mathsf{v} \to \mathsf{v}$, 3 with $\mathsf{v} \to \mathsf{v} \to \mathsf{v}$, etc. This will preserve the reduction in the previous section. So even if we only considered LFs over a small, fixed set of atomic types, repairing them with QR would remain NP-complete.

Despite being generally intractable, the instances of LF REPAIR that semanticists solve in practice tend to be easy. What could explain this discrepancy? Semanticists have most likely been working with a class of LFs subject to implicit constraints that vastly reduce the complexity of the problem. For example, the higher-order meanings that semanticists actually handle are extremely uniform in that almost all of them have a type ending in t.[5] The higher-order types appearing in the LFs from Section 4 by contrast were extremely *non*-uniform. This variety is what made it possible to construct a computationally difficult "domino game" where input and output types link together in an intricate arrangement. If this is on the right track, then the true source of complexity, the true reason why LF REPAIR is difficult, is just the enormous number

[5]This is a strong tendency, not a rule. A simple counterexample are choice functions, which have type $(et)e$.

of derivational possibilities that arise when higher-order types are variegated enough that they can interact in nontrivial ways.

A natural question for further research is whether a reduction similar to the one used in this paper could prove NP-completeness results for other scope-taking formalisms. The type-logical grammar NL_λ, for example, is a very close relative of the grammar QRT from Section 3, and should have a similar time complexity (Barker, 2019; Moot, 2020). Another good candidate are continuized CCGs (Barker and Shan, 2014; White et al., 2017).

Acknowledgments

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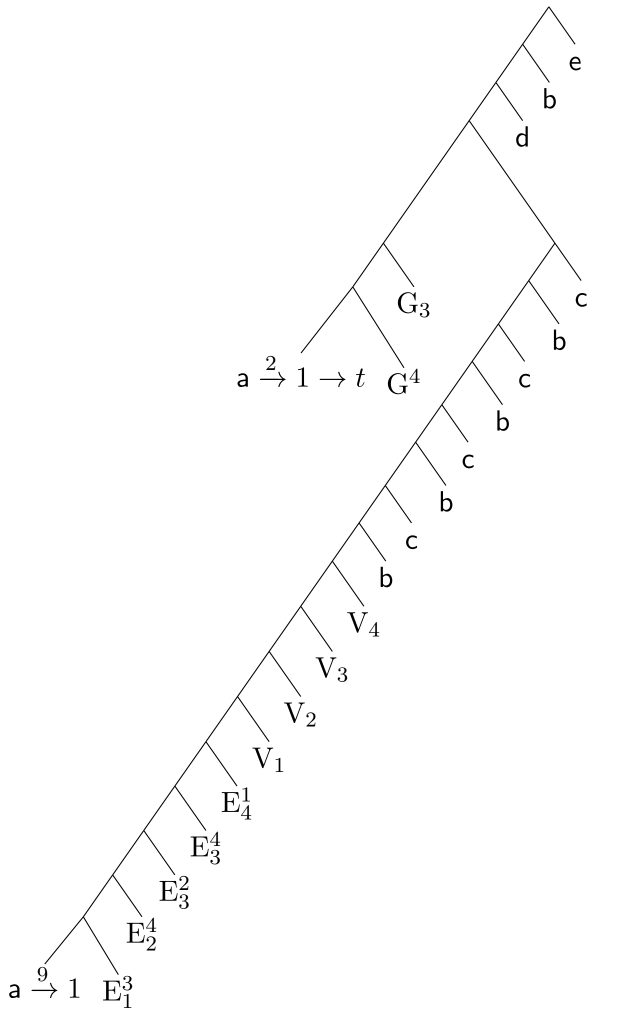

Figure 6: The LF REPAIR-instance for the digraph in Figure 4.